Theory of Many-Particle Systems. I*†

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This is the first of a series of papers dealing with many-particle systems from a unified, nonperturbative point of view. It contains derivations and discussions of various field-theoretical techniques which will be applied in subsequent papers. In a short introduction the general method of approach is summarized, and its relationship to other field-theoretic problems indicated. In the second section the macroscopic properties of the spectra of many-particle systems are described. Asymptotic evaluations are performed which characterize these macroscopic features in terms of intensive parameters, and the relationship of these parameters to thermodynamics is discussed. The special characteristics of the ground state are shown to follow as a limiting case of the asymptotic evaluations. The third section is devoted to the time-dependent field correlation functions, or Green's functions, which describe the microscopic behavior of a multiparticle system. These functions are defined, and related to intensive macroscopic variables when the energy and number of particles are large. Spectral representations and other properties of various one-particle Green's functions are derived. In the fourth section the treatment of nonequilibrium processes is considered. As a particular example, the electromagnetic properties of a system are expressed in terms of

the special two-particle Green's function which describes current correlation. The discussion yields specifically a fluctuation-dissipation theorem, a sum rule for conductivity, and certain dispersion relations. The fifth section deals with the differential equations which determine the Green's functions. The boundary conditions that characterize the Green's function equations are exhibited without reference to adiabatic decoupling. A method for solving the equations approximately, by treating the correlations among successively larger numbers of particles, is considered. The first approximation in this sequence is shown to yield a generalized Hartree-like equation. A related, but rigorous, identity for the single-particle Green's function is then derived. A second approximation, which takes certain two-particle correlations into account, is shown to produce various additional effects: The interaction between particles is altered in a manner characterized by the intensive macroscopic parameters, and the modification and spread of the energy-momentum relation come into play. In the final section compact formal expressions for the Green's functions and other physical quantities are derived. Alternative equations and systematic approximations for the Green's functions are obtained.

I. INTRODUCTION

IN any physical system the properties actually subject to measurement are few in number. These attributes, energy, momentum, angular momentum, number of particles, and the like, are distinguished by the feature that each is related to an invariance property under a simple group of transformations. In the case of energy-momentum it is the group of space-time translations, while for angular momentum, it is the rotation group. Associated with each of these groups there is a continuous unitary representation which is characterized by a set of Hermitian generators. In quantum field theory, these Hermitian operators are constructed as space integrals of products of field operators, $\psi(\mathbf{r}t)$, $\psi^\dagger(\mathbf{r}t)$. At a given time, the field operators have simple commutation properties characteristic of the particle statistics.

A many-particle system, in the context of quantum field theory, is one for which the eigenvalue of the number operator is large. In such a system, physically recognizable changes in energy are so huge compared to the energy intervals between neighboring states that the energy levels may be assumed to vary continuously. When a system is this large, the quantities of interest naturally fall into two categories. The first concerns the behavior of extensive quantities such as energy and number for which only macroscopic changes are measurable; the second refers to microscopic features involving changes in energy and number that are negligible on the macroscopic scale.

The purpose of this paper is to develop general methods for treating multiparticle systems from the quantum field-theoretical viewpoint.[1] In this discussion it will prove useful to employ and extend the mathematical techniques which were devised for application to relativistic quantum field theory. These same tech-

[1] The many-body problem has been studied with the aid of perturbation theory by many authors. This paper will not draw on any results of these works but has various points of contact with them. We mention the work of T. Matsubara, Progr. Theoret. Phys. (Kyoto) 14, 351 (1955); K. M. Watson, Phys. Rev. 103, 489 (1956); W. Riesenfeld and K. M. Watson, Phys. Rev. 104, 492 (1956), 108, 518 (1957), and other articles; K. Brueckner, Phys. Rev. 100, 36 (1955), and other articles; K. Brueckner and S. L. Gammel, Phys. Rev. 109, 1038 (1958); J. Goldstone, Proc. Roy. Soc. (London) A239, 267 (1957); L. Van Hove, Physica 22, 343 (1956), and other articles; N. M. Hugenholz, Physica 23, 481 (1957), and other articles; Huang, Lee, and Yang, Lecture at Stevens Institute Conference on Many-Body Problems, 1957 (to be published); Lee, Huang, and Yang, Phys. Rev. 106, 1135 (1957); T. D. Lee and C. N. Yang, Phys. Rev. 112, 1419 (1958), and 113, 1406 (1959); Schafroth, Butler, and Blatt, Helv. Phys. Acta 30, 93 (1957); E. Montroll and J. Ward, Phys. Fluids 1, 55 (1958); J. Hubbard, Proc. Roy. Soc. (London) A240, 539 (1957), and other articles; J. Lindhardt, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 28, No. 8 (1954); P. Nozières and D. Pines, Nuovo cimento 9, 470 (1958); A. Klein and R. Prange, Phys. Rev. 112, 994, 1008 (1958); C. DeDominicis and C. Bloch, Nuclear Phys. 7, 459 (1958), and other articles; R. H. Kraichnan, Phys. Rev. 112, 1054 (1958); S. T. Beliaev, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 417 (1958) [translation: Soviet Phys. JETP 7, 289 (1958)]. A spirit similar to ours occurs in the work of V. M. Galitskii and A. B. Migdal, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 139 (1958) [translation: Soviet Phys. JETP 7, 96 (1958)]; the note of L. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 262 (1958) [translation: Soviet Phys. JETP 7, 182 (1958)]; and in E. S. Fradkin, J. Exptl. Theoret. Phys. (U.S.S.R.) 36, 951, 1286 (1959).

niques, it will appear, achieve in a natural manner the above-mentioned separation of the macroscopic and microscopic domains. In the former domain they lend themselves to the discussion of the spectral densities of the extensive constants of motion; in the latter domain they provide a characterization of the local properties of the system. It is only the second class of problem which is usually considered in relativistic quantum field theory where the vacuum appears as the natural reference state.

The two aspects of the many-particle problem are intimately related, of course. Indeed, the most feasible method for determining the macroscopic properties of the system is by integrating the differential equations which relate them to the microscopic behavior. The quantities that fully describe the local behavior, and which thereby serve to characterize both the macroscopic and microscopic aspects of the situation, are time-dependent field correlation functions, or, in the language of field theory, Green's functions. With the aid of these functions it is possible to present in a unified manner all aspects of large systems, both static and kinetic. The information thus provided is purely dynamical, but it occurs in a form which is immediately applicable to the statistical mechanical treatment of equilibrium phenomena.

II. MACROSCOPIC PROPERTIES. DENSITY OF STATES

The spectral measure of the commuting Hermitian operators which describe the constants of motion is most conveniently found in terms of the unitary operator representation that originally determined these Hermitian operators. More specifically, Fourier transformation of the traces of the group of unitary transformations yields the desired measure. For example, if a system has a Hamiltonian, H, and a number operator, N, the formal function

$$e^{W(i\lambda, i\tau)} \equiv \mathrm{Tr}\, e^{-iN\lambda - iH\tau} \qquad (2.1)$$

generates the spectral density of the commuting quantities, energy and number. The expression (2.1) may be written in terms of P_{NE}, the projection operator for states of given energy and number, or, more explicitly, as

$$e^W = \sum_{NE} e^{-iN\lambda - iE\tau} \mathrm{Tr}\, P_{NE} \equiv \sum_{NE} e^{-iN\lambda - iE\tau}\, \mathrm{Tr}_{NE}\, 1, \qquad (2.2)$$

with the summation extending over energy eigenvalues, E, and number eigenvalues,[2] N. In terms of a formal spectral density, $\rho(NE)$, Eq. (2.2) becomes

$$e^{W(i\lambda, i\tau)} = \sum_N e^{-iN\lambda} \int dE\, e^{-iE\tau} \rho(NE). \qquad (2.3)$$

[2] The use of the same symbol for the number operator and its eigenvalue causes no difficulty in context.

From it the spectral density is determined by Fourier transformation,

$$\rho(NE) = \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{iN\lambda + iE\tau + W(i\lambda, i\tau)}. \qquad (2.4)$$

However, the summation (2.1) of an infinite number of increasingly rapidly varying exponentials is only formal and requires a summability procedure to give it meaning. The most useful way of giving a rigorous significance to expressions like (2.4) is based upon the interpretation of (2.1) as the boundary value of an analytic function in the lower half λ and τ complex planes. Since the spectra of energy and number are bounded from below, and if, as is presumably true in all physical systems, the density of states grows at most algebraically with increasing energy and number, the exponential decrease introduced by arbitrarily small negative imaginary parts of τ and λ is sufficient to guarantee absolute convergence. If, furthermore, the energy per particle increases without bound for sufficiently large particle densities, as it does in a fermion system with no attractive forces, or in any system which cannot be indefinitely compressed, the exponential factor in the energy produced by a negative imaginary part of τ assures absolute convergence whatever the imaginary part of λ.

For a large class of systems it is possible to evaluate (2.4) asymptotically by taking advantage of the analytic behavior outlined above. This is most clearly accomplished by first defining the result of the λ integration, which projects out the trace with a given number of particles,

$$\exp W_N(i\tau) \equiv \mathrm{Tr}\, P_N e^{-iH\tau} \equiv \mathrm{Tr}_N\, e^{-iH\tau}, \qquad (2.5)$$

and then evaluating asymptotically the τ integral,

$$\rho(NE) = \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \exp[iE\tau + W_N(i\tau)]. \qquad (2.6)$$

As a function of τ, $\exp W_N$ is analytic throughout the lower half-plane. Furthermore, along the negative imaginary τ axis, the second derivative of W_N with respect to $i\tau$ is real and positive, since it has the form of the dispersion of the energy with respect to a positive weight function. Consequently $-\partial W_N/\partial i\tau$, which approaches infinity as $\tau \to 0$, decreases monotonically as $\tau \to -i\infty$ to the limit given by $E_0(N)$, the minimum energy of the system for N particles. It follows that on the negative imaginary axis there is one and only one solution, $i\tau_0 = \beta > 0$, to the equation

$$E = -\partial W_N(\beta)/\partial\beta, \quad E \geqslant E_0(N). \qquad (2.7)$$

The expression (2.6) may then be integrated by the method of steepest descents, by deforming the contour of integration into the lower half plane so that it passes through β in the direction, parallel to the real axis, along which the integrand decreases most rapidly from

its maximum. The contribution to the integral from the neighborhood of this saddle point is evaluated in terms of the positive-definite second derivative

$$\partial^2 W_N/\partial\beta^2 = -(\partial E/\partial\beta)_N \geqslant 0, \tag{2.8}$$

with the result

$$\rho(NE) = \exp[\beta E + W_N(\beta)][2\pi\partial^2 W_N(\beta)/\partial\beta^2]^{-\frac{1}{2}}. \tag{2.9}$$

A more detailed justification of the asymptotic dominance of this neighborhood will not be given here.

The function, $W_N(\beta)$, is now written in terms of $W(i\lambda,\beta)$, by making explicit the steps which led to (2.6) from (2.4)

$$\exp[W_N(\beta)] = \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} e^{iN\lambda + W(i\lambda,\beta)}. \tag{2.10}$$

The integral (2.10) can also be evaluated asymptotically. Under the assumption of a maximum particle density, $\exp[W(i\lambda,\beta)]$ is analytic throughout the λ plane. An argument similar to that employed for $\partial W_N/\partial i\tau$ shows that as λ decreases along the imaginary axis $-\partial W/\partial i\lambda$ decreases monotonicaly from $\bar{N}(\beta)$, the maximum value consistent with the given β, to the value zero. Consequently there exists a unique point, $\lambda_0 = -i\alpha$, lying on the imaginary axis, which satisfies

$$N = -\partial W/\partial\alpha, \quad \bar{N}(\beta) \geqslant N \geqslant 0. \tag{2.11}$$

By deforming the contour to pass through this unique point of steepest descent along the imaginary λ axis, an asymptotic value for (2.10) is obtained,

$$\exp[W_N(\beta)] = e^{\alpha N + W(\alpha,\beta)} \left[2\pi \frac{\partial^2 W}{\partial\alpha^2} \right]^{-\frac{1}{2}}, \tag{2.12}$$

which, when inserted into (2.9) yields

$$\rho(NE) = e^{\alpha N + \beta E + W(\alpha,\beta)} \left[(2\pi)^2 \frac{\partial^2 W}{\partial\alpha^2} \frac{\partial^2 W_N}{\partial\beta^2} \right]^{-\frac{1}{2}}. \tag{2.13}$$

For the systems of physical interest, the exponential factors in (2.12) and (2.13) are the numerically significant parts and consequently (2.12) may be written

$$W_N(\beta) = \alpha N + W(\alpha,\beta). \tag{2.14}$$

This asymptotic approximation is consistent with the indicated dependence of the functions since

$$dW_N(\beta) = \alpha dN + \frac{\partial W}{\partial\beta}(\alpha,\beta)d\beta, \tag{2.15}$$

in virtue of (2.11). Equation (2.7) can therefore be replaced by

$$E = -\partial W(\alpha,\beta)/\partial\beta. \tag{2.16}$$

Furthermore, the second derivative satisfies

$$\frac{\partial^2 W_N}{\partial\beta^2} = \frac{\partial^2 W}{\partial\beta^2} + \frac{\partial^2 W}{\partial\alpha\partial\beta}\left(\frac{\partial\alpha}{\partial\beta}\right)_N$$

$$= \frac{\partial^2 W}{\partial\beta^2} - \left(\frac{\partial^2 W}{\partial\alpha\partial\beta}\right)^2 \left(\frac{\partial^2 W}{\partial\alpha^2}\right)^{-1}, \tag{2.17}$$

so that (2.13) may also be written in the form

$$\rho(NE) = e^{\alpha N + \beta E + W(\alpha,\beta)}$$

$$\times \frac{1}{2\pi}\left[\frac{\partial^2 W}{\partial\alpha^2} \frac{\partial^2 W}{\partial\beta^2} - \left(\frac{\partial^2 W}{\partial\alpha\partial\beta}\right)^2 \right]^{-\frac{1}{2}}. \tag{2.18}$$

Again, in physical systems the numerically significant parts of this expression are the exponentials, so that

$$\ln\rho(NE) = \alpha N + \beta E + W(\alpha,\beta), \tag{2.19}$$

which is consistent with the differential relation

$$d\ln\rho(NE) = \alpha dN + \beta dE. \tag{2.20}$$

The result (2.18) has been obtained in two stages. With the assurance that the dominant contribution arises from the vicinity of the one point α, β, the two-dimensional integration can be performed directly. Using this procedure, we are led to the quadratic form

$$\delta^2 W = \frac{\partial^2 W}{\partial\alpha^2}(\delta\alpha)^2 + 2\frac{\partial^2 W}{\partial\alpha\partial\beta}(\delta\alpha)(\delta\beta) + \frac{\partial^2 W}{\partial\beta^2}(\delta\beta)^2. \tag{2.21}$$

This expression is shown to be positive definite by exhibiting it as a dispersion

$$\delta^2 W = \langle (N\delta\alpha + H\delta\beta)^2 \rangle^{\alpha\beta} - (\langle N\delta\alpha + H\delta\beta\rangle^{\alpha\beta})^2, \tag{2.22}$$

with respect to the averaging process

$$\langle X \rangle^{\alpha\beta} = (\text{Tr } e^{-\alpha N - \beta H}X)/\text{Tr } e^{-\alpha N - \beta H}$$
$$= \text{Tr } e^{-\alpha N - \beta H - W(\alpha,\beta)}X. \tag{2.23}$$

The positive definiteness is fully characterized by the three inequalities

$$\partial^2 W/\partial\alpha^2 \geqslant 0, \quad \partial^2 W/\partial\beta^2 \geqslant 0,$$

$$\frac{\partial^2 W}{\partial\alpha^2}\frac{\partial^2 W}{\partial\beta^2} - \left(\frac{\partial^2 W}{\partial\alpha\partial\beta}\right)^2 = -\left(\frac{\partial E}{\partial\beta}\right)_N \frac{\partial^2 W}{\partial\alpha^2}$$

$$= -\left(\frac{\partial N}{\partial\alpha}\right)_E \frac{\partial^2 W}{\partial\beta^2} \geqslant 0. \tag{2.24}$$

By deforming the contours to pass through the unique point at which

$$N = -\partial W/\partial\alpha = \langle N\rangle^{\alpha\beta}, \quad E = -\partial W/\partial\beta = \langle H\rangle^{\alpha\beta}, \tag{2.25}$$

and integrating the quadratic form in the neighborhood of this point, we obtain the expressions (2.18) and (2.20) directly.

It is necessary at this point to describe an intensive property characteristic of any spatially distributed system—the pressure. This quantity is occasionally introduced by referring to an external parameter, the container volume, which is regarded as occurring in the Hamiltonian. The negative rate of change of the energy with respect to this volume is then used to define the pressure. It is clearly preferable, however, to introduce the notion of pressure in the more fundamental way which relates it to the local transport of momentum. Needless to say, the container walls are the agency ultimately responsible for producing the local stresses in the absence of other external forces.

We shall restrict ourselves to systems containing one type of particle described by a field, ψ, for which the momentum density operator is ($\hbar = 1$)

$$\mathbf{G}(\mathbf{r}t) = (1/2i)[\psi^\dagger(\mathbf{r}t)\nabla\psi(\mathbf{r}t) - (\nabla\psi^\dagger(\mathbf{r}t))\psi(\mathbf{r},t)]. \quad (2.26)$$

We shall also suppose, for definiteness, that the Hamiltonian has the form

$$H = \frac{1}{2m}\int d\mathbf{r}\ \nabla\psi^\dagger(\mathbf{r}t)\cdot\nabla\psi(\mathbf{r}t)$$

$$+\tfrac{1}{2}\int d\mathbf{r}d\mathbf{r}'\ \psi^\dagger(\mathbf{r}t)\psi^\dagger(\mathbf{r}'t)v(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}'t)\psi(\mathbf{r}t), \quad (2.27)$$

which is characteristic of instantaneous two-particle interactions. From the equation of motion

$$i\frac{\partial}{\partial t}\psi(\mathbf{r}t) = -\frac{1}{2m}\nabla^2\psi(\mathbf{r}t)$$

$$+\int d\mathbf{r}'\ \psi^\dagger(\mathbf{r}'t)v(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}'t)\psi(\mathbf{r}t), \quad (2.28)$$

and the adjoint equation, we then derive

$$\frac{\partial\mathbf{G}}{\partial t} = -\nabla\cdot\mathbf{T}^{(0)}$$

$$-\int d\mathbf{r}'\ \psi^\dagger(\mathbf{r}t)\psi^\dagger(\mathbf{r}'t)\nabla v(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}'t)\psi(\mathbf{r}t), \quad (2.29)$$

where the first term is the negative divergence of the stress tensor for a noninteracting system,

$$\mathbf{T}_{ij}^{(0)} = (1/2m)[\nabla_i\psi^\dagger\nabla_j\psi + \nabla_j\psi^\dagger\nabla_i\psi - \tfrac{1}{2}\nabla_i\nabla_j(\psi^\dagger\psi)], \quad (2.30)$$

while the second term of (2.29), because of the nonlocal nature of the interaction, is not a divergence. Nevertheless, for short-range forces, the second term may be replaced effectively by a divergence since the contribution to its integral over any volume comes entirely from a region in the neighborhood of the boundary of that volume whose dimension is comparable in size with the range of forces. This follows from the fact that the

double integral over identical regions for $\mathbf{r}$ and $\mathbf{r}'$ vanishes since the integrand is antisymmetrical, a property which reflects the equality of action and reaction. The effective interaction stress tensor, $\mathbf{T}^{(1)}$, is constructed by integrating the term identified with its divergence

$$\nabla\cdot\mathbf{T}^{(1)} = \int d\mathbf{r}'\ \psi^\dagger(\mathbf{r}t)\psi^\dagger(\mathbf{r}'t)\nabla v(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}'t)\psi(\mathbf{r}t), \quad (2.31)$$

over a region, V, large compared to the range of forces. In terms of $\mathbf{n}$, a unit vector normal to the surface of V, we may write

$$\int dS_1\mathbf{n}\cdot\mathbf{T}^{(1)}(\mathbf{r}_1) = \int_V d\mathbf{r}_1\int d\mathbf{r}_2\Big[\psi^\dagger(\mathbf{r}_1)\psi^\dagger(\mathbf{r}_2)$$

$$\times\frac{\mathbf{r}_{12}}{r_{12}}\frac{\partial}{\partial r_{12}}v(r_{12})\psi(\mathbf{r}_2)\psi(\mathbf{r}_1)\Big], \quad (2.32)$$

where $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ and $r_{12} = |\mathbf{r}_{12}|$. The contributions to (2.32) come entirely from a region in which $\mathbf{r}_1$ is inside V, $\mathbf{r}_2$ is outside V, and $\mathbf{r}_{12}$ is smaller than the effective range of forces.

We now introduce a hypothesis of local uniformity, asserting that within a physically small region the significant expectation values of field operator products are dependent only upon relative position vectors. The integral (2.32) may then be written in the form

$$\int dS_1\ \mathbf{n}\cdot\mathbf{T}^{(1)}(\mathbf{r}_1) = \int d\mathbf{r}_{12}\int d\mathbf{r}_2[\quad],$$

where, for fixed $\mathbf{r}_{12}$, the domain accessible to $\mathbf{r}_2$ is a shell of thickness $-\mathbf{n}\cdot\mathbf{r}_{12}$ so that $d\mathbf{r}_2 \rightarrow -\mathbf{n}\cdot\mathbf{r}_{12}dS_2$. As a result of the restriction that $\mathbf{r}_2$ be outside and $\mathbf{r}_1$ inside V, only half the $\mathbf{r}_{12}$ space is covered, and therefore

$$\mathbf{T}^{(1)}(\mathbf{r}) = -\tfrac{1}{2}\int d\mathbf{r}_{12}\mathbf{r}_{12}\mathbf{r}_{12}\ \psi^\dagger(\mathbf{r}+\tfrac{1}{2}\mathbf{r}_{12})\psi^\dagger(\mathbf{r}-\tfrac{1}{2}\mathbf{r}_{12})$$

$$\times\frac{1}{r_{12}}\frac{\partial}{\partial r_{12}}v(r_{12})\psi(\mathbf{r}-\tfrac{1}{2}\mathbf{r}_{12})\psi(\mathbf{r}+\tfrac{1}{2}\mathbf{r}_{12}). \quad (2.33)$$

The pressure, $p(\mathbf{r})$, is identified as the average diagonal element of the total stress tensor, $\tfrac{1}{3}\sum_i\mathbf{T}_{ii}$, and thus the appropriate operator becomes

$$p(\mathbf{r}) = \tfrac{1}{3}\Big[\frac{1}{m}\nabla\psi^\dagger\cdot\nabla\psi - \tfrac{1}{2}\int d\mathbf{r}_{12}\ \psi^\dagger(\mathbf{r}+\tfrac{1}{2}\mathbf{r}_{12})\psi^\dagger(\mathbf{r}-\tfrac{1}{2}\mathbf{r}_{12})$$

$$\times r_{12}\frac{\partial v(r_{12})}{\partial r_{12}}\psi(\mathbf{r}-\tfrac{1}{2}\mathbf{r}_{12})\psi(\mathbf{r}+\tfrac{1}{2}\mathbf{r}_{12})\Big]. \quad (2.34)$$

The assertion of local uniformity has again been invoked to justify the omission of the term containing $\nabla^2(\psi^\dagger\psi)$.

We can now make contact with the viewpoint that relates the pressure to the total volume occupied by the system. For this purpose we consider the trace definition of the function $W(\alpha,\beta,V)$, and perform the coordinate scale transformation

$$\mathbf{r}=\theta\mathbf{r}', \qquad (2.35)$$

together with the operator canonical transformation,

$$\psi(\mathbf{r})=\theta^{-\frac{3}{2}}\psi'(\mathbf{r}'). \qquad (2.36)$$

In view of the invariance of the trace under a canonical transformation we find that with the new scale of coordinates

$$\exp[W(\alpha,\beta,\theta^3 V)]$$

$$=\mathrm{Tr}\exp\left[-\alpha N-\beta\theta^{-2}\int\frac{1}{2m}\nabla\psi^\dagger\cdot\nabla\psi\right.$$

$$\left.-\tfrac{1}{2}\beta\int\psi^\dagger\psi^\dagger v(\theta r)\psi\psi\right]. \qquad (2.37)$$

Differentiation with respect to θ at the point $\theta=1$ now yields

$$V\frac{\partial W}{\partial V}=\beta\left\langle\int d\mathbf{r}\,p(\mathbf{r})\right\rangle^{\alpha\beta}=\beta pV, \qquad (2.38)$$

and the numerical pressure so defined is given by

$$p=(1/\beta)(\partial W/\partial V). \qquad (2.39)$$

Furthermore, for the system being considered, which involves no external forces, the operator $p(\mathbf{r})$ describes a local property independent of the total volume, so that the pressure is an intensive variable, $p(\alpha,\beta)$. Hence $W(\alpha,\beta,V)$ must have the form

$$W=V\beta p(\alpha,\beta), \qquad (2.40)$$

and likewise the energy and particle densities are given by the intensive expressions

$$\frac{N}{V}=-\frac{\partial}{\partial\alpha}(\beta p), \quad \frac{E}{V}=-\frac{\partial}{\partial\beta}(\beta p). \qquad (2.41)$$

For the purpose of introducing a conventional notation we note that the relation

$$dW=-Nd\alpha-Ed\beta+\beta pdV \qquad (2.42)$$

may be written as

$$d(W+\alpha N+\beta E)=\alpha dN+\beta dE+\beta pdV,$$

or as the following extension of Eq. (2.20):

$$(1/\beta)d\ln\rho=(\alpha/\beta)dN+dE+pdV. \qquad (2.43)$$

We also note that (2.40) and (2.19) lead directly to

$$(1/\beta)\ln\rho=(\alpha/\beta)N+E+pV. \qquad (2.44)$$

It is clear that thermodynamic systems may be discussed by interpreting the symbol $1/\beta$ as the absolute temperature (multiplied by k), $\ln\rho$ as the entropy, S (divided by k), and $-\alpha/\beta$ as the chemical potential, μ. We shall for convenience use this thermodynamic language without, in any way, confining ourselves to domain of equilibrium processes to which thermodynamics applies. The inequality (2.24) may be conveniently reexpressed in terms of the thermodynamic variable μ. In particular since the chemical potential is given by

$$\mu=(\partial E/\partial N)_{SV}, \qquad (2.45)$$

its derivative satisfies the relation

$$\left(\frac{\partial^2 E}{\partial N^2}\right)_{SV}=\left(\frac{\partial\mu}{\partial N}\right)_{SV}=-\frac{1}{\beta}\left(\frac{\partial\alpha}{\partial N}\right)_{SV}+\frac{\alpha}{\beta^2}\left(\frac{\partial\beta}{\partial N}\right)_{SV} \qquad (2.46)$$

$$=-\frac{1}{\beta(dN)^2}[(d\alpha)(dN)+(d\beta)(dE)] \qquad (2.47)$$

$$=\frac{1}{\beta}\frac{1}{(dN)^2}d^2W\geqslant 0, \qquad (2.48)$$

which may also be written in the form

$$\left(\frac{\partial}{\partial N}(E-\mu N)\right)_{SV}\leqslant 0. \qquad (2.49)$$

In a homogeneous system, for which the energy density is an intensive variable and $p=-(\partial E/\partial V)_{SN}$, the relation (2.48) may be written in the form

$$-\left(\frac{\partial p}{\partial V}\right)_{SN}=\left(\frac{\partial^2 E}{\partial V^2}\right)_{SN}=\left(\frac{N}{V}\right)^2\left(\frac{\partial^2 E}{\partial N^2}\right)_{SV}\geqslant 0. \qquad (2.50)$$

It also follows from (2.24) that

$$(\partial\mu/\partial N)_{\beta V}\geqslant 0, \qquad (2.51)$$

which in a homogeneous system reduces to

$$-\left(\frac{\partial p}{\partial V}\right)_{\beta N}=\left(\frac{N}{V}\right)^2\left(\frac{\partial\mu}{\partial N}\right)_{\beta V}\geqslant 0. \qquad (2.52)$$

One basic problem to which the techniques developed in the remainder of this paper shall be applied in the determination of the ground-state properties of an interacting system. Such a state does not strictly fall within the realm of the preceding asymptotic evaluations. Yet the properties of this particular state are obtained by taking the zero-temperature limit of the expressions derived above. In particular we note first that in the limit as $\beta\to\infty$, $\ln\rho$, which satisfies

$$\left(\frac{\partial\ln\rho}{\partial\beta}\right)_{NV}=\beta\left(\frac{\partial E}{\partial\beta}\right)_{NV}\leqslant 0, \qquad (2.53)$$

approaches its minimum value, zero, and E approaches its minimum, $E_0(N)$, both of which correctly characterize the ground state. In this limit Eq. (2.44) reduces to

$$\mu N = pV + E_0(N). \tag{2.54}$$

The conditions (2.49) and (2.51), which are equivalent for the ground state, may be applied to (2.54) where they require that

$$\left(\frac{\partial p}{\partial N}\right)_V = -\frac{1}{V}\left(\frac{\partial}{\partial N}(E_0(N) - \mu N)\right)_V \geqslant 0. \tag{2.55}$$

Furthermore since $p = 0$ when $N = 0$ and p is continuous,

$$p = -(1/V)(E_0(N) - \mu N) \geqslant 0. \tag{2.56}$$

A particularly simple situation occurs when the ground state of the system is spatially localized by its own forces and requires no container. For such a state the pressure satisfies

$$p = -(\partial E_0/\partial V)_N = 0, \tag{2.57}$$

so that the asymptotic equation (2.54) reduces to

$$E_0 = \mu N. \tag{2.58}$$

Turning from the asymptotic evaluation to the direct discussion of the ground state we note that these same results are obtained. In this case the essential point is that, in a large system interacting by short-range forces, the energy per particle in the ground state is a function of the density only. From the stationary property of the ground-state energy with respect to the total occupied volume, which requires that

$$(\partial E_0/\partial V)_N = 0, \tag{2.59}$$

it then follows, in agreement with (2.58), that

$$\frac{V}{N}\left(\frac{\partial E_0}{\partial V}\right)_N = -\left(\frac{\partial E_0}{\partial N}\right)_V + \frac{E_0}{N} = 0. \tag{2.60}$$

Similar considerations may be applied to unbound systems by including the work done by the container walls.

III. MICROSCOPIC PROPERTIES. THE GREEN'S FUNCTIONS

The microscopic properties of a system which are of physical interest are those which are essentially identical in a large fraction of the states with prescribed values for the extensive constants of the motion. For this reason, it must be possible to determine the properties by averaging over all states with the prescribed constants. In the present section we shall be concerned with a class of average values in terms of which all such quantities may be expressed. It is the set of average disturbances induced by the introduction of external particle sources localized in space and time.

In the previous section it was demonstrated that the density of states as a function of energy and number could always be determined from the function $W(i\lambda, i\tau)$. Indeed, in a many-particle system, the macroscopic properties at each level of energy and number were shown to depend upon $W(\alpha, \beta)$ for unique real α and β determined by E and N. As a result the system was characterized by intensive parameters denoted as temperature and chemical potential, from which, conversely, the energy and number might be found.

In the determination of the class of averages which describes the microscopic properties, procedures analogous to those of the previous section will be followed. First a rigorous method for obtaining the averages, independent of asymptotic considerations, will be indicated. Then an asymptotic evaluation of these quantities will be carried out. In this asymptotic evaluation we shall find that the relevant macroscopic variables are the intensive quantities, α and β.

The class of Green's functions associated with prescribed macroscopic properties N and E are the set of average diagonal matrix elements

$$\begin{aligned} G_n{}^{NE}(&\mathbf{r}_1 t_1 \cdots \mathbf{r}_n t_n; \mathbf{r}_1' t_1' \cdots \mathbf{r}_n' t_n') \\ &= (-i)^n \epsilon [NE | (\psi(\mathbf{r}_1 t_1) \cdots \psi(\mathbf{r}_n t_n) \\ &\quad \times \psi^\dagger(\mathbf{r}_n' t_n') \cdots \psi^\dagger(\mathbf{r}_1' t_1'))_+ | NE], \quad (3.1) \end{aligned}$$

where

$$\begin{aligned} [NE | X | NE] &\equiv \mathrm{Tr}(P_{NE}X)/\mathrm{Tr}\, P_{NE} \\ &= \mathrm{Tr}_{NE}X/\mathrm{Tr}_{NE}1. \end{aligned} \tag{3.2}$$

The structure of these Green's functions is related to the dynamical description in terms of external disturbances; consequently the operators ψ and $\psi^\dagger$ occur from right to left in the order of increasing time coordinates. [This ordering is denoted by the ( )$_+$ bracket.] The factor ϵ in (3.1) is identically $+1$ for boson fields while for fermions it is the antisymmetrical function of the time coordinates, which equals $+1$ when the operators occur in the order exhibited in (3.1). The subscript n refers to the number of field operators ψ or $\psi^\dagger$ which occur in the function. Since the field operators create and destroy particles, the number n represents the maximum difference in the number of particles between various intermediate states and the original state with N particles. We shall refer to (3.1) as the n-particle Green's function although, as just indicated, the name actually refers to the number of particles superimposed on the background. Finally, the coordinates $\mathbf{r}$ should be understood to contain implicitly the internal coordinates, like spins, which may also be present.

As with the density of states, it is convenient to determine the function (3.1) by introducing a generating Green's function characterized by τ and λ, which in a notation generalized from (2.23),

$$\langle X \rangle^{i\lambda, i\tau} = [\mathrm{Tr}(e^{-iN\lambda - iH\tau}X)]/\mathrm{Tr}\, e^{-iN\lambda - iH\tau}, \tag{3.3}$$

has the form

$$G_n{}^{i\lambda,\,i\tau}=(-i)^n\epsilon\langle(\psi(\mathbf{r}_1t_1)\cdots\psi(\mathbf{r}_nt_n)$$
$$\times\psi^\dagger(\mathbf{r}_n't_n')\cdots\psi^\dagger(\mathbf{r}_1't_1'))_+\rangle^{i\lambda,\,i\tau}. \quad (3.4)$$

These functions, $G^{i\lambda,\,i\tau}$, are defined as quotients of traces so that the dominant rapid variation, associated with the exponentials of the energy and number of the entire system, is removed. From them, the functions (3.1) are determined by the relation

$$\rho(NE)G^{NE}=\int\frac{d\lambda}{2\pi}\int\frac{d\tau}{2\pi}\,e^{iN\lambda+iE\tau+W(i\lambda,\,i\tau)}G^{i\lambda,\,i\tau}. \quad (3.5)$$

While W and the Green's functions have been independently defined it should be noted that the derivatives of W are simply related to the Green's functions. The λ derivative of W satisfies

$$-\frac{\partial W}{\partial(i\lambda)}=\langle N\rangle^{i\lambda,\,i\tau} \quad (3.6)$$

$$=\pm i\int d\mathbf{r}\,G_1{}^{i\lambda,\,i\tau}(\mathbf{r}t;\mathbf{r}t^+), \quad (3.7)$$

where t is any time, and t^+ indicates the ordering of the field operators at equal times. The alternative signs, $+$ and $-$, refer, respectively, to bosons (B.E.) and fermions (F.D.). Similarly, the τ derivative of W satisfies

$$-\frac{\partial W}{\partial(i\tau)}=\langle H\rangle^{i\lambda,\,i\tau}, \quad (3.8)$$

so that when the Hamiltonian has the form (2.27), we may write

$$-\frac{\partial W}{\partial(i\tau)}=\pm i\int d\mathbf{r}\,\lim_{\mathbf{r}'\to\mathbf{r}}-\frac{\nabla^2}{2m}G_1{}^{i\lambda,\,i\tau}(\mathbf{r}t;\mathbf{r}'t^+)$$
$$-\tfrac{1}{2}\int d\mathbf{r}_1 d\mathbf{r}_2\,v(\mathbf{r}_1-\mathbf{r}_2)G_2{}^{i\lambda,\,i\tau}(\mathbf{r}_1t\mathbf{r}_2t;\mathbf{r}_1t^+\mathbf{r}_2t^+). \quad (3.9)$$

With the aid of the field equation (2.28),

$$\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}\right)\psi(\mathbf{r}t)$$
$$-\int d\mathbf{r}'\,v(\mathbf{r}'-\mathbf{r})\psi^\dagger(\mathbf{r}'t)\psi(\mathbf{r}'t)\psi(\mathbf{r}t)=0, \quad (3.10)$$

the relation (3.9) may be written in an alternative form involving only G_1, namely

$$-\frac{\partial W}{\partial i\tau}=\pm i\int d\mathbf{r}\,\lim_{t'\to t^+,\,\mathbf{r}'\to\mathbf{r}}\tfrac{1}{2}\left(i\frac{\partial}{\partial t}-\frac{\nabla^2}{2m}\right)G_1{}^{i\lambda,\,i\tau}(\mathbf{r}t;\mathbf{r}'t').$$
$$(3.11)$$

Considerable information about the structure of the functions $G^{i\lambda,\,i\tau}$ is obtained from their definition as traces and from the commutation relations for the field operators. For example, the one-particle function $G_1{}^{i\lambda,\,i\tau}$ for $t>t'$ is equal to the function

$$G_>{}^{i\lambda,\,i\tau}(\mathbf{r}t;\mathbf{r}'t')=e^{-W(i\lambda,\,i\tau)}$$
$$\times\mathrm{Tr}[e^{-iN\lambda-iH\tau}(1/i)\psi(\mathbf{r}t)\psi^\dagger(\mathbf{r}'t')], \quad (3.12)$$

and for $t<t'$, it agrees with

$$G_<{}^{i\lambda,\,i\tau}(\mathbf{r}t;\mathbf{r}'t')=\pm e^{-W(i\lambda,\,i\tau)}$$
$$\times\mathrm{Tr}[e^{-iN\lambda-iH\tau}(1/i)\psi^\dagger(\mathbf{r}'t')\psi(\mathbf{r}t)]. \quad (3.13)$$

From the cyclic properties of the trace, it follows that (3.13) is equivalent to

$$G_<{}^{i\lambda,\,i\tau}(\mathbf{r}t;\mathbf{r}'t')=\pm e^{-W(i\lambda,\,i\tau)}$$
$$\times\mathrm{Tr}[(1/i)\psi(\mathbf{r}t)e^{-iN\lambda-iH\tau}\psi^\dagger(\mathbf{r}'t')]. \quad (3.14)$$

The evolution of the field operators in time is governed by the relation

$$\psi(\mathbf{r}t_1)=e^{iH(t_1-t_2)}\psi(\mathbf{r}t_2)e^{-iH(t_1-t_2)}, \quad (3.15)$$

and the annihilation properties of the field operators imply that

$$\psi f(N)=f(N+1)\psi. \quad (3.16)$$

In virtue of these relations, Eq. (3.14) may be rewritten as

$$G_<{}^{i\lambda,\,i\tau}(\mathbf{r}t;\mathbf{r}'t')=\pm e^{-W}\,\mathrm{Tr}[e^{-i\lambda}e^{-iN\lambda-iH\tau}(1/i)$$
$$\times\psi(\mathbf{r},t+\tau)\psi^\dagger(\mathbf{r}'t')] \quad (3.17)$$

$$=\pm e^{-i\lambda}G_>(\mathbf{r}t+\tau;\mathbf{r}'t'). \quad (3.18)$$

If Fourier transforms of these functions which depend only on $t-t'$ are introduced,

$$G_{>,<}{}^{i\lambda,\,i\tau}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}e^{-i\omega(t-t')}g_{>,<}{}^{i\lambda,\,i\tau}(\mathbf{r}\mathbf{r}'\omega), \quad (3.19)$$

then it follows from (3.18) that

$$g_<{}^{i\lambda,\,i\tau}(\mathbf{r}\mathbf{r}'\omega)=\pm e^{-i\lambda-i\omega\tau}g_>{}^{i\lambda,\,i\tau}(\mathbf{r}\mathbf{r}'\omega). \quad (3.20)$$

Further information may be obtained from the commutation relations, which require that

$$i[G(\mathbf{r}t;\mathbf{r}'t^-)-G(\mathbf{r}t;\mathbf{r}'t^+)]=\delta(\mathbf{r}-\mathbf{r}'). \quad (3.21)$$

Since the values of these Green's functions are given by

$$G_>(\mathbf{r}t;\mathbf{r}'t)=\int\frac{d\omega}{2\pi i}g_>(\mathbf{r}\mathbf{r}'\omega)$$

and

$$G_<(\mathbf{r}t;\mathbf{r}'t)=\int\frac{d\omega}{2\pi i}g_<(\mathbf{r}\mathbf{r}'\omega), \quad (3.22)$$

the functions $g_>$ and $g_<$ satisfy

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}[g_> - g_<] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}[1 \mp e^{-i\lambda - i\omega\tau}]g_> = \delta(\mathbf{r} - \mathbf{r}').$$
(3.23)

These formulas may be written in a more symmetrical form by introducing the function A according to

$$g_>^{i\lambda,i\tau}(\mathbf{r}\mathbf{r}'\omega) = \frac{A^{i\lambda,i\tau}(\mathbf{r}\mathbf{r}'\omega)}{1 \mp e^{-i\lambda - i\omega\tau}};$$

$$g_<^{i\lambda,i\tau}(\mathbf{r}\mathbf{r}'\omega) = -\frac{A^{i\lambda,i\tau}(\mathbf{r}\mathbf{r}'\omega)}{1 \mp e^{i\lambda + i\omega\tau}}.$$
(3.24)

This function A satisfies

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\mathbf{r}\mathbf{r}'\omega) = \delta(\mathbf{r} - \mathbf{r}'),$$
(3.25)

and in terms of it, the Green's function may be written

$$G^{i\lambda,i\tau}(\mathbf{r}t;\mathbf{r}'t') = \frac{1}{i} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{A^{i\lambda,i\tau}(\mathbf{r}\mathbf{r}'\omega)}{1 \mp e^{-i\lambda - i\omega\tau}}, \quad t > t'$$
(3.26)

$$= -\frac{1}{i} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{A^{i\lambda,i\tau}(\mathbf{r}\mathbf{r}'\omega)}{1 \mp e^{i\lambda + i\omega\tau}}, \quad t < t'.$$
(3.27)

To obtain a single expression for the Green's function, we write

$$G(\mathbf{r}t;\mathbf{r}'t') = \eta_+(t-t')G_>(\mathbf{r}t;\mathbf{r}'t')$$
$$+ \eta_-(t-t')G_<(\mathbf{r}t;\mathbf{r}'t'), \quad (3.28)$$

where

$$\eta_+(t-t') = 1, \quad t > t'$$
$$= 0, \quad t < t';$$
$$\eta_-(t-t') = 0, \quad t > t'$$
$$= 1, \quad t < t',$$
(3.29)

and employ the integral representation

$$\eta_\pm(t-t') = \pm i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega \pm i\epsilon}, \quad (\epsilon \to 0^+) \quad (3.30)$$

together with the evaluation

$$\frac{1}{\omega \pm i\epsilon} = P\frac{1}{\omega} \mp \pi i\delta(\omega).$$
(3.31)

The result is

$$G(\mathbf{r}t;\mathbf{r}'t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\left[P\frac{1}{\omega-\omega'}\right.$$

$$\left. - \pi i\delta(\omega-\omega')\frac{1 \pm e^{-i\lambda - i\omega\tau}}{1 \mp e^{-i\lambda - i\omega\tau}}\right]A^{i\lambda,i\tau}(\mathbf{r}\mathbf{r}'\omega'). \quad (3.32)$$

So far in this section no asymptotic considerations have been employed. With the aid of such considerations we shall now show that G^{NE} reduces asymptotically to $G^{i\lambda,i\tau}$ with $i\lambda$ and $i\tau$ replaced by the values of α and β appropriate to the given N and E. As was remarked after (3.4), the function $G^{i\lambda,i\tau}$ is the quotient of two terms possessing the same rapid variation, but in which the numerator also contains functions that vary only to the extent of exponential factors involving the energy and number of n excitations. These factors are of negligible importance compared to those involving E and N and so the asymptotic evaluation of (3.5) is essentially identical with that of (2.4). The only difference lies in the fact that at the saddle point α, β, the integrand is now multiplied by the slowly varying function evaluated at that point,

$$G^{\alpha\beta} \equiv G^{i\lambda,i\tau}, \quad i\lambda = \alpha, \ i\tau = \beta. \quad (3.33)$$

Equation (3.5) therefore yields the asymptotic identity

$$G^{NE} \cong G^{\alpha\beta}.$$
(3.34)

We note here the forms assumed by the derivatives of $W(\alpha,\beta)$, as derived from the previously obtained relations (3.7), (3.9), and (3.11):

$$-\frac{\partial W}{\partial \alpha} = \langle N \rangle^{\alpha\beta} = N = \pm i \int d\mathbf{r}\, G_1^{\alpha\beta}(\mathbf{r}t;\mathbf{r}t^+), \quad (3.35)$$

$$-\frac{\partial W}{\partial \beta} = \langle H \rangle^{\alpha\beta} = E = \pm i \int d\mathbf{r} \lim_{\mathbf{r}' \to \mathbf{r}}\left[-\frac{\nabla^2}{2m}G_1^{\alpha\beta}(\mathbf{r}t;\mathbf{r}'t^+)\right]$$

$$-\tfrac{1}{2}\int d\mathbf{r}d\mathbf{r}'\, v(\mathbf{r}-\mathbf{r}')G_2^{\alpha\beta}(\mathbf{r}t\mathbf{r}'t; \mathbf{r}t^+\mathbf{r}'t^+), \quad (3.36)$$

$$= \pm i \int d\mathbf{r} \lim_{t' \to t^+, \mathbf{r}' \to \mathbf{r}}\left[\frac{1}{2}\left(i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m}\right)G_1^{\alpha\beta}(\mathbf{r}t;\mathbf{r}'t')\right]. \quad (3.37)$$

It is instructive to analyze the function G_1^{NE} rigorously and to demonstrate how it reduces asymptotically to the Green's function $G_1^{\alpha\beta}$. We begin with the definition (3.1) which relates the Green's function $G_1^{NE}(\mathbf{r}t;\mathbf{r}'t')$ for $t > t'$ to the function

$$G_>^{NE}(\mathbf{r}t;\mathbf{r}'t') = (1/i)[NE|\psi(\mathbf{r}t)\psi^\dagger(\mathbf{r}'t')|NE]. \quad (3.38)$$

As a matrix in the coordinate indices $\mathbf{r}t$ and $\mathbf{r}'t'$, the function $iG_>^{NE}$ is Hermitian,

$$iG_>^{NE}(\mathbf{r}t;\mathbf{r}'t') = [iG_>^{NE}(\mathbf{r}'t';\mathbf{r}t)]^*, \quad (3.39)$$

and positive definite. We shall use the notation

$$iG_>^{NE}(\mathbf{r}t;\mathbf{r}'t') \geqslant 0 \quad (3.40)$$

to indicate this matrix property. No confusion will result from this notation since, when it does refer to numbers, these will be the elements of a diagonal

matrix. Equation (3.38) may be written in the form

$$iG_>^{NE}(\mathbf{r}t; \mathbf{r}'t')$$

$$= \sum_{N'E'} [NE|\psi(\mathbf{r}t)P_{N'E'}\psi^\dagger(\mathbf{r}'t')|NE]. \quad (3.41)$$

If we take into account the creation and annihilation properties of the field operators and the operator description of their time dependence, we may reduce (3.41) to

$$iG_>^{NE}(\mathbf{r}t; \mathbf{r}'t') = [\mathrm{Tr}_{NE}1]^{-1} \sum_{E'} \mathrm{Tr}[P_{NE}\psi(\mathbf{r})$$

$$\times P_{N+1, E'}\psi^\dagger(\mathbf{r}')]e^{-i(E'-E)(t-t')}. \quad (3.42)$$

In this equation, the arbitrary common time of the field operators has been omitted.

The Fourier transform of this expression is defined as

$$G_>^{NE}(\mathbf{r}t; \mathbf{r}'t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} e^{-i\omega(t-t')} g_>^{NE}(\mathbf{r}\mathbf{r}'\omega), \quad (3.43)$$

where

$$g_>^{NE}(\mathbf{r}\mathbf{r}'\omega) = 2\pi\rho(N+1, E+\omega)$$
$$\times \mathrm{Tr}[P_{NE}\psi(\mathbf{r})P_{N+1, E+\omega}\psi^\dagger(\mathbf{r}')]/$$
$$(\mathrm{Tr}_{NE}1)(\mathrm{Tr}_{N+1, E+\omega}1). \quad (3.44)$$

The transform, $g_>^{NE}$, has properties as a matrix in $\mathbf{r}$ and $\mathbf{r}'$ which correspond to the conditions (3.39) and (3.40),

$$g_>^{NE}(\mathbf{r}\mathbf{r}'\omega) = g_>^{NE}(\mathbf{r}'\mathbf{r}\omega)^* \geqslant 0. \quad (3.45)$$

Furthermore, its form indicates the special properties which apply to the ground state of the system with N particles, for which $E = E_0(N)$. The energy $E' = E_0(N) + \omega$ must be a possible energy for the system with $N+1$ particles, so that

$$\omega = E(N+1) - E_0(N) \geqslant E_0(N+1) - E_0(N)$$
$$\cong \partial E_0(N)/\partial N \quad (3.46)$$

and only frequencies greater than the chemical potential, $\mu = \partial E_0(N)/\partial N$, appear.

Similarly, the Green's function for $t < t'$ satisfies the relations

$$iG_<^{NE}(\mathbf{r}t; \mathbf{r}'t) = [iG_<^{NE}(\mathbf{r}'t'; \mathbf{r}t)]^*, \quad (3.47)$$

$$\pm iG_<^{NE}(\mathbf{r}t; \mathbf{r}'t') \geqslant 0. \quad (3.48)$$

If its Fourier transform is defined by

$$G_<^{NE}(\mathbf{r}t; \mathbf{r}'t') = \int \frac{d\omega}{2\pi i} e^{-i\omega(t-t')} g_<^{NE}(\mathbf{r}\mathbf{r}'\omega), \quad (3.49)$$

the Hermitian amplitude

$$\pm g_<^{NE}(\mathbf{r}\mathbf{r}'\omega) = 2\pi\rho(N-1, E-\omega)$$
$$\times \mathrm{Tr}[P_{NE}\psi^\dagger(\mathbf{r}')P_{N-1, E-\omega}\psi(\mathbf{r})]/$$
$$(\mathrm{Tr}_{NE}1)(\mathrm{Tr}_{N-1, E-\omega}1) \quad (3.50)$$

is also positive definite. For the ground state we infer a frequency condition converse to (3.46); $E' = E_0(N) - \omega$ is an energy for $N-1$ particles and consequently

$$\omega = E_0(N) - E(N-1) \leqslant E_0(N) - E_0(N-1)$$
$$\cong \partial E_0(N)/\partial N = \mu. \quad (3.51)$$

As Eqs. (3.46) and (3.51) suggest, a certain simplicity can frequently be achieved by adding a term, $-\mu_0 N$, to the Hamiltonian, which displaces the zero of energy for the individual particles. Then conditions (3.46) and (3.51) become restrictions to positive and negative frequencies for positive and negative time differences, respectively.

Thus far the function G_1^{NE} has been analyzed exactly, with the exception of the replacement

$$E_0(N+1) - E_0(N) \cong \partial E_0(N)/\partial N.$$

To proceed further, however, it is useful to involve the asymptotic properties in a more fundamental way to derive a parallel to the exact relation (3.20). We note that when many particles are present, the averaged matrix elements will be fractionally insensitive to a change of one in the total number, or to a change in the total energy by an amount associated with a single excitation. Equation (3.50) may then be replaced by

$$\pm g_<^{NE}(\mathbf{r}\mathbf{r}'\omega) \cong 2\pi\rho(NE)$$
$$\times \mathrm{Tr}[P_{N+1, E+\omega}\psi^\dagger(\mathbf{r}')P_{NE}\psi(\mathbf{r})]/$$
$$(\mathrm{Tr}_{N+1, E+\omega}1)(\mathrm{Tr}_{NE}1) \quad (3.52)$$

$$\cong g_>^{NE}(\mathbf{r}\mathbf{r}'\omega)[\rho(NE)/\rho(N+1, E+\omega)]. \quad (3.53)$$

The ratio of the density of states at infinitesimally different energies, for systems differing by one in the number of particles, is given by Eq. (2.43),

$$d \ln\rho = \alpha dN + \beta dE \rightarrow \alpha + \beta\omega. \quad (3.54)$$

Consequently Eq. (3.53) coincides with the form of (3.20) obtained by the replacement $i\lambda \rightarrow \alpha$, $i\tau \rightarrow \beta$:

$$g_<(\mathbf{r}\mathbf{r}'\omega) = \pm e^{-\alpha - \beta\omega} g_>(\mathbf{r}\mathbf{r}'\omega). \quad (3.55)$$

We note that the conditions (3.46) and (3.51), rigorously valid for the ground state, are reproduced by these asymptotic representations as $\beta \rightarrow \infty$ with $-\alpha/\beta = \mu$ held fixed. The subsequent analysis proceeds in exact parallel with (3.21) through (3.32) where the function $A^{\alpha\beta}$, defined by

$$g_>(\mathbf{r}\mathbf{r}'\omega) = \frac{A^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega)}{1 \mp e^{-\alpha - \beta\omega}}, \quad g_<(\mathbf{r}\mathbf{r}'\omega) = -\frac{A^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega)}{1 \mp e^{\alpha + \beta\omega}}, \quad (3.56)$$

satisfies

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega) = \delta(\mathbf{r} - \mathbf{r}'), \quad (3.57)$$

and occurs in the Green's functions in the form

$$G^{NE}(\mathbf{r}t; \mathbf{r}'t') \cong G^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t')$$

$$= \frac{1}{i} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{A^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega)}{1 \mp e^{-\alpha-\beta\omega}}, \quad t > t' \qquad (3.58)$$

$$= -\frac{1}{i} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{A^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega)}{1 \mp e^{\alpha+\beta\omega}}, \quad t < t' \qquad (3.59)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left\{ \mathrm{P}\frac{1}{\omega-\omega'} - \pi i\delta(\omega-\omega') \right.$$

$$\left. \times \{\tanh[\tfrac{1}{2}\beta(\omega-\mu)]\}^{\mp 1} \right\} A^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega'). \quad (3.60)$$

The positive-definite matrix conditions $g_>{}^{NE} \geqslant 0$ and $\pm g_<{}^{NE} \geqslant 0$ imply that

$$A^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega) \coth[\tfrac{1}{2}\beta(\omega-\mu)] \geqslant 0, \quad \text{B.E.,}$$
$$A^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega) \geqslant 0, \quad \text{F.D.} \qquad (3.61)$$

For a large class of systems it is possible to simplify the spatial dependence of the function $G_1(\mathbf{r}t; \mathbf{r}'t') = G_1(\mathbf{r}\mathbf{r}'; t-t')$ by using the fact that in the interior of a homogeneous medium the field correlation functions depend only upon coordinate differences,

$$G_1(\mathbf{r}\mathbf{r}'; t-t') \cong G_1(\mathbf{r}-\mathbf{r}'; t-t'). \qquad (3.62)$$

It is convenient to transform to momentum space by introducing the Fourier representation

$$G_1(\mathbf{r}-\mathbf{r}'; t-t') = \int \frac{d\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')} G(\mathbf{p}; t-t'). \qquad (3.63)$$

The momentum-space Green's function is then given by [for definiteness we use the asymptotic forms although identical formulas obtain for the functions $G_1{}^{i\lambda,i\tau}(\mathbf{p}; t-t')$]

$$G_1{}^{\alpha\beta}(\mathbf{p}; t-t') = \frac{1}{i} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{A^{\alpha\beta}(\mathbf{p}\omega)}{1 \mp e^{-\alpha-\beta\omega}}, \quad t > t'$$

$$= -\frac{1}{i} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{A^{\alpha\beta}(\mathbf{p}\omega)}{1 \mp e^{\alpha+\beta\omega}}, \quad t < t' \qquad (3.64)$$

where the amplitude $A^{\alpha\beta}(\mathbf{p}\omega)$, which is still a matrix in the internal variables, obeys the additional relations

$$\int \frac{d\omega}{2\pi} A^{\alpha\beta}(\mathbf{p}\omega) = 1,$$

$$A^{\alpha\beta}(\mathbf{p}\omega) \coth[\tfrac{1}{2}(\alpha+\beta\omega)] \geqslant 0, \quad \text{B.E.,} \qquad (3.65)$$

$$A^{\alpha\beta}(\mathbf{p}\omega) \geqslant 0, \quad \text{F.D.}$$

Equation (3.35) then reduces to

$$N = \int d\mathbf{r} \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \frac{\mathrm{tr}\, A^{\alpha\beta}(\mathbf{p}\omega)}{e^{\alpha+\beta\omega} \mp 1}, \qquad (3.66)$$

or in terms of the occupied volume,

$$\frac{N}{V} = \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \frac{\mathrm{tr}\, A^{\alpha\beta}(\mathbf{p}\omega)}{e^{\alpha+\beta\omega} \mp 1}. \qquad (3.67)$$

In these expressions summation of diagonal elements of the internal coordinate matrices has been explicitly denoted by the symbol tr. We may evidently identify the frequency integral with the momentum distribution per unit volume of phase space,

$$n(\mathbf{p}) = \int \frac{d\omega}{2\pi} \frac{\mathrm{tr}\, A^{\alpha\beta}(\mathbf{p}\omega)}{e^{\alpha+\beta\omega} \mp 1}. \qquad (3.68)$$

In a similar manner, Eq. (3.37) becomes

$$\frac{E}{V} = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{2}\left(\omega + \frac{\mathbf{p}^2}{2m}\right) \frac{\mathrm{tr}\, A^{\alpha\beta}(\mathbf{p}\omega)}{e^{\alpha+\beta\omega} \mp 1}. \qquad (3.69)$$

The equality of the second derivatives,

$$\frac{\partial E}{\partial \alpha} = \frac{\partial^2 W}{\partial \alpha \partial \beta} = \frac{\partial N}{\partial \beta}, \qquad (3.70)$$

imposes an additional condition on the numbers $\mathrm{tr}\, A^{\alpha\beta}(\mathbf{p}\omega)$, namely

$$\int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \frac{1}{2}\left(\frac{\mathbf{p}^2}{2m} - \omega\right) \frac{e^{\alpha+\beta\omega}\, \mathrm{tr}\, A^{\alpha\beta}(\mathbf{p}\omega)}{(e^{\alpha+\beta\omega} \mp 1)^2}$$

$$= \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \frac{1}{e^{\alpha+\beta\omega} \mp 1}\left[-\frac{\partial}{\partial\beta} \mathrm{tr}\, A^{\alpha\beta}(\mathbf{p}\omega)\right.$$

$$\left. + \frac{1}{2}\left(\frac{\mathbf{p}^2}{2m} + \omega\right) \frac{\partial}{\partial\alpha} \mathrm{tr}\, A^{\alpha\beta}(\mathbf{p}\omega)\right]. \qquad (3.71)$$

For noninteracting particles of either statistics the function $A^{\alpha\beta}(\mathbf{p}\omega)$ has the particularly simple form[3]

$$A^{\alpha\beta}(\mathbf{p}\omega) = 2\pi\delta(\omega - \mathbf{p}^2/2m). \qquad (3.72)$$

With this function, Eqs. (3.67) to (3.69) reduce to familiar statements, and the integrability condition (3.71) is explicitly satisfied. We also infer from (3.65) the special restriction associated with noninteracting bosons, $\alpha \geqslant 0$ or $\mu \leqslant 0$.

IV. TWO-PARTICLE GREEN'S FUNCTIONS. ELECTROMAGNETIC PROPERTIES

While it is possible to discuss some macroscopic properties and the behavior of a certain class of "single-

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[3] We defer the proof of this expected result to the later more general discussion.

particle" excitations in terms of $G_1(\mathbf{r}t; \mathbf{r}'t')$, a treatment of the transport properties of the system is most easily carried out in terms of multiparticle Green's functions. Typically, the transport of the quantity of interest results from the application of an external field to a system in which there is no flux initially. The flux induced by the disturbing field may then be computed in two distinct ways. The first involves exact solution of the field equations in the presence of the external agency; the second treats the disturbance as a perturbation of the system which is still described by the eigenstate basis appropriate to the unperturbed Hamiltonian. The quantities whose transport properties are of interest (energy, momentum, matter, charge, and the like), are all characterized by multilinear forms of field operators and their derivatives. The flux of these quantities is consequently described perturbatively by means of expectation values of products of these multilinear operator expressions at different space-time points. Such expectation values comprise a special subset of Green's functions in which some of the coordinate arguments are set equal. In the present section we shall illustrate this perturbation treatment of transport phenomena by a detailed consideration of one simple example, the flow of electric charge. Since the electromagnetic field is coupled with the charge and current densities, consideration of this problem leads in particular to the discussion of the special two-particle Green's function,

$$F_{kl}{}^{NE}(\mathbf{r}t; \mathbf{r}'t') = [NE|(\mathbf{j}_k(\mathbf{r}t)\mathbf{j}_l(\mathbf{r}'t'))_+|NE], \quad (4.1)$$

where $\mathbf{j}$ is the current density operator in the absence of the applied field. The corresponding correlation function for the charge density does not require explicit discussion since its coupling can be removed by working in the gauge in which the scalar potential vanishes. The gauge invariance which permits this choice is of course intimately tied to the current conservation relation, by means of which, conversely, the behavior of the charge may be determined from that of the current.

In the presence of an external field, described in the gauge in which the scalar potential vanishes, the kinetic energy term of the Hamiltonian is altered from that of (2.27) to

$$H = \frac{1}{2m}\int d\mathbf{r}\left(-\frac{1}{i}\mathbf{\nabla}-\frac{e\mathbf{A}}{c}\right)\psi^\dagger(\mathbf{r})\left(\frac{1}{i}\mathbf{\nabla}-\frac{e\mathbf{A}}{c}\right)\psi(\mathbf{r}). \quad (4.2)$$

(In the case of fermions there are also effects resulting from the spin magnetic moment which we shall omit.) The current density is the coefficient of the first variation in $-(1/c)\mathbf{A}$ and is therefore equal to

$$\mathbf{J} = \frac{e}{2m}\left[-\frac{1}{i}(\mathbf{\nabla}\psi^\dagger)\psi+\psi^\dagger\frac{1}{i}\mathbf{\nabla}\psi-\frac{2e\mathbf{A}}{c}\psi^\dagger\psi\right] \equiv \mathbf{J}^{(0)}+\mathbf{J}^{(1)}. \quad (4.3)$$

In this expression the operators ψ are solutions of the field equations in the presence of the electromagnetic field. The addition to the Hamiltonian, $H^{(A)}$, resulting from the appearance of the vector potential[4] in (4.2) is

$$H^{(A)} = -\frac{1}{c}\int \mathbf{A}\cdot\mathbf{J}^{(0)}-\frac{1}{2c}\int \mathbf{A}\cdot\mathbf{J}^{(1)}. \quad (4.4)$$

To determine the linear induced electromagnetic properties of the medium it is necessary to evaluate, to first order in the external field, the average current

$$[NE|\mathbf{J}^{(0)}|NE]+[NE|\mathbf{J}^{(1)}|NE] \quad (4.5)$$

generated in those states which had energy E and number N before the field was applied. The current $[NE|\mathbf{J}^{(1)}|NE]$ contains this field linearly and is therefore known to the required order. The current $[NE|\mathbf{J}^{(0)}|NE]$ in the presence of the field may be determined to the desired accuracy by inserting the first approximation to the unitary transformation that gives the accumulated effect of the external field for the interval between t_0 and t, during which the electromagnetic field acts,

$$[NE|\mathbf{J}^{(0)}(\mathbf{r}t)|NE]$$

$$\cong \left[NE\left|\left(1+i\int_{t_0}^t H^{(A)}(t')dt'\right)\mathbf{j}(\mathbf{r}t)\right.\right.$$

$$\left.\left.\times\left(1-i\int_{t_0}^t H^{(A)}(t')dt'\right)\right|NE\right]. \quad (4.6)$$

Accordingly, this part of the induced current is

$$[NE|\mathbf{J}^{(0)}(\mathbf{r}t)|NE]$$

$$\cong \left[NE\left|\left[\mathbf{j}(\mathbf{r}t), \frac{i}{c}\int_{t_0}^t dt'\int d\mathbf{r}'\,\mathbf{j}(\mathbf{r}'t')\cdot\mathbf{A}(\mathbf{r}'t')\right]\right|NE\right]. \quad (4.7)$$

Since the field can be represented by a vector potential $\mathbf{A}(\mathbf{r}t)$ which is identically zero for $t<t_0$, we may write

$$[NE|\mathbf{J}^{(0)}(\mathbf{r}t)|NE] = \frac{i}{c}\int_{-\infty}^t dt'\int d\mathbf{r}'$$

$$\times[NE|[\mathbf{j}(\mathbf{r}t), \mathbf{j}(\mathbf{r}'t')]|NE]\cdot\mathbf{A}(\mathbf{r}'t'), \quad (4.8)$$

and then combine the two terms of (4.5) into

$$[NE|\mathbf{J}(\mathbf{r}t)|NE] = \int_{-\infty}^{\infty} d\mathbf{r}'dt'\,\kappa(\mathbf{r}t; \mathbf{r}'t')\cdot\frac{1}{c}\mathbf{A}(\mathbf{r}'t'), \quad (4.9)$$

where the polarization tensor, κ, is expressed in terms of the states and operators in the absence of the external

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[4] In this discussion it is the external potential that is designated by $\mathbf{A}$. The electromagnetic interactions among particles are understood to be included in the unperturbed Hamiltonian.

field,

$$\kappa_{kl}(\mathbf{r}t; \mathbf{r}'t')$$
$$= \eta(t-t')i[NE|[\mathbf{j}_k(\mathbf{r}t),\mathbf{j}_l(\mathbf{r}'t')]|NE]$$
$$- \delta_{kl}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')(e^2/m)$$
$$\times[NE|\psi^\dagger(\mathbf{r}t)\psi(\mathbf{r}t)|NE]. \quad (4.10)$$

Here $\eta(t-t')$ is the function

$$\eta(t-t') = \eta_+(t-t') \quad (4.11)$$

defined in (3.29). The asymptotic considerations which led to (3.33) apply equally well to the commutator and permit the replacement

$$\kappa \cong \eta(t-t')i\langle[\mathbf{j}_k(\mathbf{r}t),\mathbf{j}_l(\mathbf{r}'t')]\rangle^{\alpha\beta} - \delta_{kl}\delta(\mathbf{r}-\mathbf{r}')$$
$$\times \delta(t-t')(e^2/m)\langle\psi^\dagger(\mathbf{r}t)\psi(\mathbf{r}t)\rangle^{\alpha\beta}. \quad (4.12)$$

Although the current correlation functions occurring in (4.1), (4.10), and (4.12) are determined from two-particle Green's functions, their space-time dependence is that of a single-particle function of a "current field," which in addition possesses symmetries consequent to the Hermitian character of this field. Rather than follow the example of Sec. III and first describe the current correlations with the aid of the functions $F_{kl}{}^{i\lambda,i\tau}(\mathbf{r}t; \mathbf{r}'t')$ and $F_{kl}{}^{NE}(\mathbf{r}t; \mathbf{r}'t')$, we turn this time directly to the function $F_{kl}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t')$. Similar manipulations would appear in a direct analysis of $G_1{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t')$.

When $t > t'$, the function $F_{kl}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t')$ is equal to

$$F_{kl>}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t')$$
$$= \mathrm{Tr}[e^{-\alpha N - \beta H - W(\alpha,\beta)}\mathbf{j}_k(\mathbf{r}t)\mathbf{j}_l(\mathbf{r}'t')], \quad (4.13)$$

while for $t < t'$, it is represented by

$$F_{kl<}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t')$$
$$= \mathrm{Tr}[e^{-\alpha N - \beta H - W(\alpha,\beta)}\mathbf{j}_l(\mathbf{r}'t')\mathbf{j}_k(\mathbf{r}t)]. \quad (4.14)$$

As matrices in the indices $k,\mathbf{r},t$ and $l,\mathbf{r}',t'$, these functions are each Hermitian,

$$F_{kl>,<}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t') = F_{lk>,<}{}^{\alpha\beta}(\mathbf{r}'t'; \mathbf{r}t)^*, \quad (4.15)$$

and positive definite

$$F_{kl}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t') \geqslant 0. \quad (4.16)$$

Since the two functions $F_>$ and $F_<$ (unlike $G_>$ and $G_<$) are really the same function evaluated at different arguments, we have the connection

$$F_{kl<}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t') = F_{lk>}{}^{\alpha\beta}(\mathbf{r}'t'; \mathbf{r}t), \quad (4.17)$$

which in virtue of (4.15) becomes

$$F_{kl<}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t') = F_{kl>}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t')^*. \quad (4.18)$$

The relation exhibiting the trace structure, which is analogous to (3.18), can be written formally as

$$F_{kl<}{}^{\alpha\beta}(\mathbf{r}\mathbf{r}'; t-t') = F_{kl>}{}^{\alpha\beta}(\mathbf{r}\mathbf{r}'; t-t'-i\beta). \quad (4.19)$$

The Fourier transforms are introduced here as

$$F_{kl>,<}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f_{kl>,<}{}^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega)e^{-i\omega(t-t')}. \quad (4.20)$$

In consequence of the properties (4.15) to (4.19), we have the corresponding statements

$$f_{kl>,<}{}^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega) = f_{lk>,<}{}^{\alpha\beta}(\mathbf{r}'\mathbf{r}\omega)^*, \quad (4.21)$$

$$f_{kl>,<}{}^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega) \geqslant 0, \quad (4.22)$$

$$f_{kl<}{}^{\alpha\beta}(\mathbf{r}\mathbf{r}'\omega) = f_{lk>}{}^{\alpha\beta}(\mathbf{r}'\mathbf{r}-\omega) \quad (4.23)$$

$$= f_{kl>}{}^{\alpha\beta}(\mathbf{r}\mathbf{r}'-\omega)^*, \quad (4.24)$$

and

$$f_{kl<}(\mathbf{r}\mathbf{r}'\omega) = e^{-\omega\beta}f_{kl>}(\mathbf{r}\mathbf{r}'\omega). \quad (4.25)$$

These relations are most conveniently presented by introducing the real positive-definite combination, the symmetrical product

$$S_{kl}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t') = \langle\{\mathbf{j}_k(\mathbf{r}t),\mathbf{j}_l(\mathbf{r}'t')\}\rangle^{\alpha\beta} \quad (4.26)$$

$$= F_{kl>}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t') + F_{kl<}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t'). \quad (4.27)$$

Its Fourier transform, defined by

$$S_{kl}(\mathbf{r}\mathbf{r}'; t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}s_{kl}(\mathbf{r}\mathbf{r}'\omega), \quad (4.28)$$

satisfies the condition

$$s_{kl}(\mathbf{r}\mathbf{r}'\omega) = s_{kl}(\mathbf{r}'\mathbf{r}-\omega) \geqslant 0, \quad (4.29)$$

because S is symmetric and positive definite, and the condition

$$s_{kl}(\mathbf{r}\mathbf{r}'\omega) = s_{kl}(\mathbf{r}\mathbf{r}'-\omega)^*, \quad (4.30)$$

because S is real. From (4.20), (4.25), (4.27), and (4.28) we see that

$$s_{kl}(\mathbf{r}\mathbf{r}'\omega) = f_{kl>}(\mathbf{r}\mathbf{r}'\omega) + f_{kl<}(\mathbf{r}\mathbf{r}'\omega)$$
$$= (1+e^{-\omega\beta})f_{kl>}(\mathbf{r}\mathbf{r}'\omega) \quad (4.31)$$
$$= (1+e^{\omega\beta})f_{lk<}(\mathbf{r}\mathbf{r}'\omega).$$

Hence the requirements on $f_{>,<}$ are satisfied in terms of those for s and conversely. We also observe that

$$\langle[\mathbf{j}_k(\mathbf{r}t),\mathbf{j}_l(\mathbf{r}'t')]\rangle^{\alpha\beta}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}[f_{kl>}(\mathbf{r}\mathbf{r}'\omega) - f_{kl<}(\mathbf{r}\mathbf{r}'\omega)], \quad (4.32)$$

which, in view of (4.31), becomes

$$= \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')}[\tanh(\beta\omega/2)]s_{kl}(\mathbf{r}\mathbf{r}'\omega). \quad (4.33)$$

We shall restrict ourselves in the remainder of this discussion to current correlation functions appropriate to a homogeneous, isotropic medium. It is then possible, as in Sec. III, to transform to momentum space by introducing

$$s_{kl}(\mathbf{r}\mathbf{r}'\omega) = s_{kl}(\mathbf{r}-\mathbf{r}', \omega) = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}s_{kl}(\mathbf{k}\omega), \quad (4.34)$$

where $s_{kl}(\mathbf{k},\omega)$ satisfies the analogs of (4.29) and (4.30),

$$s_{kl}(\mathbf{k}\omega) = s_{lk}(-\mathbf{k}-\omega) \qquad (4.35)$$

$$= s_{lk}(\mathbf{k}\omega)^* \geqslant 0. \qquad (4.36)$$

According to our assumption, the tensor character of $s_{kl}(\mathbf{k}\omega)$ must be expressed by

$$s_{kl}(\mathbf{k}\omega) = \delta_{kl}s_1(k^2\omega^2) + (k_k k_l - k^2\delta_{kl})s_2(k^2\omega^2). \qquad (4.37)$$

In writing (4.37), we have also observed that s_{kl}, as an even function of $\mathbf{k}$, symmetric in the indices, k and l, must be an even function of ω. Furthermore $s_{kl}(\mathbf{k}\omega)$ must be real and positive definite so that

$$s_1 \geqslant 0, \quad s_1 - k^2 s_2 \geqslant 0. \qquad (4.38)$$

An additional condition on s_{kl}, imposed by the commutation relations at equal times, will be found subsequently.

We can now evaluate the polarization tensor by substituting the expression (4.33) for the commutator which occurs on the right-hand side of (4.12). With the form (4.37) appropriate to a homogeneous isotropic medium, this commutator becomes

$$\langle [\mathbf{j}_k(\mathbf{r}t), \mathbf{j}_l(\mathbf{r}'t')] \rangle$$

$$= \delta_{kl} i \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}$$

$$\times \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{\tanh(\frac{1}{2}\beta\omega)}{\omega} s_1(k^2\omega^2)$$

$$+ (\delta_{kl}\nabla^2 - \nabla_k\nabla_l) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}$$

$$\times \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} [\tanh(\frac{1}{2}\beta\omega)] s_2(k^2\omega^2). \qquad (4.39)$$

Equation (4.39) may be written in the alternative form

$$\kappa_{kl}(\mathbf{r}t; \mathbf{r}'t')$$

$$= -\frac{\partial}{\partial t}\Big[\delta_{kl}\eta(t-t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}$$

$$\times \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{\tanh(\frac{1}{2}\beta\omega)}{\omega} s_1(k^2\omega^2) \Big]$$

$$+ (\delta_{kl}\nabla^2 - \nabla_k\nabla_l)\eta(t-t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}$$

$$\times \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} i[\tanh(\frac{1}{2}\beta\omega)] s_2(k^2\omega^2)$$

$$+ \delta(t-t')\delta_{kl} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$\times \Big(\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\tanh(\frac{1}{2}\beta\omega)}{\omega} s_1(k^2\omega^2) - \frac{ne^2}{m} \Big), \qquad (4.40)$$

where n is the density of particles per unit volume. The last term in this expression vanishes in virtue of the following identities. From electric current conservation we obtain

$$\nabla \cdot \langle [\mathbf{j}(\mathbf{r}t), \mathbf{j}(\mathbf{r}'t')] \rangle = -\Big\langle \Big[\frac{\partial\rho(\mathbf{r}t)}{\partial t}, \mathbf{j}(\mathbf{r}'t') \Big] \Big\rangle. \qquad (4.41)$$

Equation (4.39) permits the replacement of the left-hand side of (4.41) by

$$i\frac{\partial}{\partial t}\nabla \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{\tanh(\frac{1}{2}\beta\omega)}{\omega} s_1(k^2\omega^2), \qquad (4.42)$$

and we thereby infer that

$$\langle [\rho(\mathbf{r},t), \mathbf{j}(\mathbf{r}'t')] \rangle^{\alpha\beta}$$

$$= -i\nabla \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$\times \frac{\tanh(\frac{1}{2}\beta\omega)}{\omega} s_1(k^2\omega^2). \qquad (4.43)$$

When $t=t'$, the commutation properties of charge and current density permit the evaluation

$$-i\nabla \frac{e^2}{m}\delta(\mathbf{r}-\mathbf{r}')n = -i\nabla \int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$\times \frac{\tanh(\frac{1}{2}\beta\omega)}{\omega} s_1(k^2\omega^2), \qquad (4.44)$$

and, finally we arrive at the relation referred to after (4.39), and required for eliminating the last term of (4.40),

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\tanh(\frac{1}{2}\beta\omega)}{\omega} s_1(k^2\omega^2) = \frac{ne^2}{m}. \qquad (4.45)$$

We may now integrate (4.9) by parts with the expression (4.40) substituted for κ. In the gauge we have utilized, the fields satisfy

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t},$$

$$\nabla \times \mathbf{H} = \nabla \times (\nabla \times \mathbf{A}) = [\nabla\nabla - \nabla^2] \cdot \mathbf{A}. \qquad (4.46)$$

We thereby obtain

$$\langle \mathbf{J}(\mathbf{r}t) \rangle = \int d\mathbf{r}' \int dt' \, 2\eta(t-t')[\sigma(\mathbf{r}-\mathbf{r}', t-t')\mathbf{E}(\mathbf{r}'t')$$

$$+ c(\partial/\partial t)\gamma(\mathbf{r}-\mathbf{r}', t-t')\nabla' \times \mathbf{H}(\mathbf{r}'t')], \qquad (4.47)$$

where $\sigma(k^2\omega^2)$ and $\gamma(k^2\omega^2)$ are the functions

$$\sigma(k^2\omega^2) = \frac{1}{2}\frac{\tanh(\frac{1}{2}\beta\omega)}{\omega}s_1(k^2\omega^2), \qquad (4.48)$$

$$c^2\gamma(k^2\omega^2) = \frac{1}{2}\frac{\tanh(\frac{1}{2}\beta\omega)}{\omega}s_2(k^2\omega^2), \qquad (4.49)$$

and

$$\sigma(\mathbf{r}-\mathbf{r}', t-t') = \int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\omega(t-t')}$$
$$\times e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\sigma(k^2\omega^2),$$
$$\gamma(\mathbf{r}-\mathbf{r}', t-t') = \int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-i\omega(t-t')}$$
$$\times e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\gamma(k^2\omega^2). \qquad (4.50)$$

These functions satisfy the positive-definiteness conditions (4.38) which assert that

$$\sigma(k^2\omega^2) \geqslant 0, \quad \sigma(k^2\omega^2) \geqslant (kc)^2\gamma(k^2\omega^2). \qquad (4.51)$$

The function σ also obeys the sum rule (4.45),

$$\frac{2}{\pi}\int_0^\infty d\omega\, \sigma(k^2\omega^2) = \frac{ne^2}{m}, \qquad (4.52)$$

which in turn implies the inequality

$$\frac{2}{\pi}\int_0^\infty d\omega\, \gamma(k^2\omega^2) \leqslant \frac{ne^2}{mc^2}\frac{1}{k^2}. \qquad (4.53)$$

With the definitions

$$\tilde{\sigma}(\mathbf{r}-\mathbf{r}', t-t') = 2\eta(t-t')\sigma(\mathbf{r}-\mathbf{r}', t-t'), \qquad (4.54)$$

$$\tilde{\chi}(\mathbf{r}-\mathbf{r}', t-t') = 2\eta(t-t')(\partial/\partial t)\gamma(\mathbf{r}-\mathbf{r}', t-t'), \qquad (4.55)$$

Eq. (4.47) may alternatively be written in the form

$$\langle \mathbf{J}(\mathbf{r}t) \rangle = \int d\mathbf{r}'dt'[\tilde{\sigma}(\mathbf{r}-\mathbf{r}', t-t')\mathbf{E}(\mathbf{r}'t')$$
$$+\tilde{\chi}(\mathbf{r}-\mathbf{r}', t-t')c\nabla'\times\mathbf{H}(\mathbf{r}'t')]. \qquad (4.56)$$

In view of the integral representation (3.30) for $\eta(t-t')$ the Fourier transform of the right side of (4.54) may be separated into its resistive part, the conductivity,[5] and reactive part, proportional to the polarizability,

$$\tilde{\sigma}(k\omega) = \frac{i}{\pi}\int_{-\infty}^\infty d\omega' \frac{\sigma(k^2\omega'^2)}{\omega-\omega'+i\epsilon} = \sigma(k^2\omega^2) - i\omega\alpha(k^2\omega^2), \qquad (4.57)$$

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[5] The connection between conductivity and pair-correlation function have been discussed in work of R. Kubo, Can. J. Phys. **34**, 1274 (1956), and J. Phys. Soc. Japan **12**, 570 (1957); W. Kohn and J. M. Luttinger, Phys. Rev. **108**, 590 (1957); H. Nakano, Progr. Theoret. Phys. (Kyoto) **15**, 77 (1956); M. Lax, Phys. Rev. **109**, 1921 (1958); S. F. Edwards, Phil. Mag. **33**, 1020 (1958).

which are related by

$$\alpha(k^2\omega^2) = \frac{2}{\pi}P\int_0^\infty d\omega' \frac{\sigma(k^2\omega'^2)}{\omega'^2-\omega^2}. \qquad (4.58)$$

From the sum rule (4.52) we conclude that in the high-frequency limit the polarizability reduces to its value for a system of free charges

$$\lim_{\omega\to\infty} \omega^2\alpha(k^2\omega^2) = -\frac{ne^2}{m}. \qquad (4.59)$$

Similarly, from (4.55) and (3.30) we obtain the expression

$$\tilde{\chi}(k\omega) = \int_{-\infty}^\infty \frac{d\omega'}{\pi}\omega'\frac{\gamma(k^2\omega'^2)}{\omega-\omega'+i\epsilon} = \chi(k^2\omega^2) - i\omega\gamma(k^2\omega^2) \qquad (4.60)$$

for the generalized magnetic susceptibility, from which, the relation

$$\chi(k^2\omega^2) = -\frac{2}{\pi}P\int_0^\infty \frac{d\omega'}{\omega'^2-\omega^2}\omega'^2\gamma(k^2\omega'^2), \qquad (4.61)$$

between its resistive and reactive parts, follows. From (4.61) and (4.53) we also infer that the zero frequency susceptibility is limited in its degree of diamagnetism,

$$\chi(k^20) \geqslant -\frac{ne^2}{mc^2}\frac{1}{k^2}, \qquad (4.62)$$

which limit becomes arbitrarily large as the wave number approaches zero.

We may now construct an integral representation for the current correlation function $F_{kl}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t')$ in a homogeneous medium in terms of the quantities of direct physical significance, σ and γ:

$$F_{kl}{}^{\alpha\beta}(\mathbf{r}t; \mathbf{r}'t') = \eta_+F_> + \eta_-F_<$$
$$= \int_{-\infty}^\infty \frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int \frac{d\mathbf{k}}{(2\pi)^3}$$
$$\times e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}f_{kl}{}^{\alpha\beta}(k\omega), \qquad (4.63)$$

$$f_{kl}{}^{\alpha\beta}(k\omega) = \int_{-\infty}^\infty \frac{d\omega'}{2\pi}\left[P\frac{1}{\omega-\omega'} - \pi i\coth(\tfrac{1}{2}\beta\omega)\delta(\omega-\omega')\right]$$
$$\times 2i\omega'[\sigma(k^2\omega'^2)\delta_{kl} + (k_kk_l - \delta_{kl}k^2)c^2\gamma(k^2\omega'^2)], \qquad (4.64)$$

which is similar in structure to the single-particle boson Green's function (3.60). Equation (4.64) may also be written in the form

$$f_{kl}{}^{\alpha\beta}(k\omega)$$
$$= [\omega\sigma(k^2\omega^2)\coth(\tfrac{1}{2}\beta\omega) - i\omega^2(\alpha(k^2\omega^2) + (ne^2/m\omega^2))]\delta_{kl}$$
$$+ [\gamma(k^2\omega^2)\omega\coth(\tfrac{1}{2}\beta\omega) + i\chi(k^2\omega^2)]c^2(k_kk_l - \delta_{kl}k^2). \qquad (4.65)$$

The real part of this expression may, through (4.26), be interpreted as the fluctuation-dissipation theorem,[6]

$$\tfrac{1}{2}\langle\{j_k(\mathbf{r}t),j_l(\mathbf{r}'t')\}\rangle^{\alpha\beta}$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\big[\sigma(\mathbf{k}^2\omega^2)\delta_{kl}$$

$$+(\mathbf{k}_k\mathbf{k}_l-\mathbf{k}^2\delta_{kl})c^2\gamma(\mathbf{k}^2\omega^2)\big]2\omega\bigg[\frac{1}{2}+\frac{1}{e^{\beta\omega}-1}\bigg]. \quad (4.66)$$

The method for calculating F and thus determining the physical properties associated with its weight function are based upon the connection of F with a two-particle function. This relation follows directly from the definitions (4.3), (4.1), and (3.1) for the current density, the current correlation function, and the Green's function:

$$F_{kl}{}^{\alpha\beta}(\mathbf{r}_1t;\,\mathbf{r}_2t')$$

$$= \lim_{\mathbf{r}_1'\to\mathbf{r}_1,\,\mathbf{r}_2'\to\mathbf{r}_2}-\bigg(\frac{e}{2m}\bigg)^2(\boldsymbol{\nabla}_1-\boldsymbol{\nabla}_{1'})_k(\boldsymbol{\nabla}_2-\boldsymbol{\nabla}_{2'})_l$$

$$\times\langle(\psi^\dagger(\mathbf{r}_1't^+)\psi(\mathbf{r}_1t)\psi^\dagger(\mathbf{r}_2't'^+)\psi(\mathbf{r}_2t'))_+\rangle \quad (4.67)$$

$$=+\frac{e^2}{(2m)^2}\lim_{\mathbf{r}_1'\to\mathbf{r}_1,\,\mathbf{r}_2'\to\mathbf{r}_2}(\boldsymbol{\nabla}_1-\boldsymbol{\nabla}_{1'})_k(\boldsymbol{\nabla}_2-\boldsymbol{\nabla}_{2'})_l$$

$$\times G_2{}^{\alpha\beta}(\mathbf{r}_1t\mathbf{r}_2t';\,\mathbf{r}_1't^+\mathbf{r}_2't'^+). \quad (4.68)$$

As we have noted, the condition of gauge invariance and current conservation permit a determination of the charge density correlation function from the current correlation function:

$$\boldsymbol{\nabla}'\boldsymbol{\nabla}:\langle\mathbf{j}(\mathbf{r}t)\mathbf{j}(\mathbf{r}'t')\rangle=\frac{\partial}{\partial t}\frac{\partial}{\partial t'}\langle\rho(\mathbf{r}t)\rho(\mathbf{r}'t')\rangle, \quad (4.69)$$

$$\boldsymbol{\nabla}'\boldsymbol{\nabla}:\langle(\mathbf{j}(\mathbf{r}t)\mathbf{j}(\mathbf{r}'t'))_+\rangle=\frac{\partial}{\partial t}\frac{\partial}{\partial t'}\langle(\rho(\mathbf{r}t)\rho(\mathbf{r}'t'))_+\rangle$$
$$-\delta(t-t')\bigg\langle\bigg[\rho(\mathbf{r}t),\frac{\partial}{\partial t}\rho(\mathbf{r}'t)\bigg]\bigg\rangle. \quad (4.70)$$

From these relations the structure of the density correlation function may be inferred. Indeed, manipulations of this character have been employed in obtaining (4.43) from which we may with one further step derive a representation for the charge density commutator[7]:

$$\langle[\rho(\mathbf{r}t),\rho(\mathbf{r}'t')]\rangle$$

$$=\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\frac{2\mathbf{k}^2}{\omega}\sigma(\mathbf{k}^2\omega^2). \quad (4.71)$$

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[6] Fluctuation-dissipation theorems are historically associated with Nyquist and have been recently discussed by H. B. Callen and T. R. Welton, Phys. Rev. **83**, 34 (1951); J. Weber, Phys. Rev. **101**, 1620 (1956).

[7] The division by ω is not valid if σ contains a delta function of ω.

By applying (4.70) to (4.65) we obtain the relation

$$\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\mathbf{k}^2[\omega\sigma\coth(\tfrac{1}{2}\beta\omega)-i\omega^2\alpha]$$

$$=\frac{\partial}{\partial t}\frac{\partial}{\partial t'}\langle(\rho(\mathbf{r}t)\rho(\mathbf{r}'t'))_+\rangle. \quad (4.72)$$

By integration, we then obtain the analog of (4.71) for the time-ordered product of charge densities,

$$\langle(\rho(\mathbf{r}t)\rho(\mathbf{r}'t'))_+\rangle-(ne)^2$$

$$=-i\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$\times\mathbf{k}^2\bigg[\alpha+i\sigma\frac{\coth(\tfrac{1}{2}\beta\omega)}{\omega}\bigg]. \quad (4.73)$$

V. DETERMINATION OF THE GREEN'S FUNCTIONS

In previous sections we have indicated how physical properties can be expressed in terms of Green's functions. We turn now to the problem of constructing the functions themselves. Of the several Green's functions previously defined, the ones most conveniently determined are the generating Green's functions, $G_n{}^{i\lambda,i\tau}$. In this section we will derive a set of equations for them, and discuss one method of approximate solution. The set we shall derive consists of integro-differential equations coupling the various multiparticle functions. For a variety of systems at low densities or with weak or slowly varying interactions, solutions to the first few equations yield Green's functions which satisfactorily describe various static and kinetic properties.

The set of differential equations for the Green's functions is generated directly from the equations of motion and commutation properties of the field operators. Thus, the equation of motion

$$i(\partial\psi(\mathbf{r}t)/\partial t)=-(\boldsymbol{\nabla}^2/2m)\psi(\mathbf{r}t)+\int d\mathbf{r}'\,v(\mathbf{r}-\mathbf{r}')$$

$$\times\psi^\dagger(\mathbf{r}'t)\psi(\mathbf{r}'t)\psi(\mathbf{r}t), \quad (5.1)$$

implies that[8]

$$[i(\partial/\partial t_1)+(\boldsymbol{\nabla}_1{}^2/2m)](-i)\epsilon(\psi(\mathbf{r}_1t_1)\psi^\dagger(\mathbf{r}_1't_1'))_+$$

$$+i\int d\mathbf{r}_2\,v(\mathbf{r}_1-\mathbf{r}_2)\epsilon(\psi^\dagger(\mathbf{r}_2t_1)\psi(\mathbf{r}_2t_1)\psi(\mathbf{r}_1t_1)\psi^\dagger(\mathbf{r}_1't_1'))_+$$

$$=[\psi(\mathbf{r}_1t_1)\psi^\dagger(\mathbf{r}_1't_1')\mp\psi^\dagger(\mathbf{r}_1't_1')\psi(\mathbf{r}_1t_1)]\delta(t_1-t_1')$$

$$=\delta(\mathbf{r}_1-\mathbf{r}_1')\delta(t_1-t_1'). \quad (5.2)$$

On multiplying this equation by the operator

$$\exp(-iN\lambda-iH\tau),$$

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[8] In the remainder of this paper we continue the omission of spin and other internal degrees of freedom which we instituted in Sec. III.

and taking the trace, we obtain, with the definition (3.4)

$$[i(\partial/\partial t_1)+(\nabla_1{}^2/2m)]G_1{}^{i\lambda,\,i\tau}(\mathbf{r}_1t_1;\,\mathbf{r}_1't_1')$$

$$\mp i\int d\mathbf{r}_2\,v(\mathbf{r}_1-\mathbf{r}_2)G_2{}^{i\lambda,\,i\tau}(\mathbf{r}_1t_1\mathbf{r}_2t_1;\,\mathbf{r}_1't_1'\mathbf{r}_2t_1{}^+)$$
$$=\delta(\mathbf{r}_1-\mathbf{r}_1')\delta(t_1-t_1').\quad(5.3)$$

Similar equations are derived by applying the differential operator of (5.2) to the multiparticle Green's functions,

$$[i(\partial/\partial t_1)+(\nabla_1{}^2/2m)]G_n{}^{i\lambda,\,i\tau}(\mathbf{r}_1t_1\cdots\mathbf{r}_nt_n;\,\mathbf{r}_1't_1'\cdots\mathbf{r}_n't_n')$$

$$\mp i\int d\mathbf{r}_{n+1}v(\mathbf{r}_1-\mathbf{r}_{n+1})$$
$$\times G_{n+1}{}^{i\lambda,\,i\tau}(\mathbf{r}_1t_1\cdots\mathbf{r}_{n+1}t_1;\,\mathbf{r}_1't_1'\cdots\mathbf{r}_{n+1}t_1{}^+)$$
$$=\sum_{i=1}^{n}\delta(\mathbf{r}_1-\mathbf{r}_i')\delta(t_1-t_i')(\pm1)^{i-1}$$
$$\times G_{n-1}{}^{i\lambda,\,i\tau}(\mathbf{r}_2t_2\cdots\mathbf{r}_nt_n;\,\mathbf{r}_1't_1'\cdots\mathbf{r}_{i-1}'t_{i-1}'$$
$$\times\mathbf{r}_{i+1}'t_{i+1}'\cdots\mathbf{r}_n't_n').\quad(5.4)$$

Corresponding differential equations describe the dependence of these functions on the space-time coordinates of the $\psi^\dagger$ field.

Since the same equations are satisfied by *any* appropriately normalized matrix element of the time-ordered field operators, it is necessary to adjoin boundary conditions to characterize the desired solution. It is convenient, for this purpose, to discuss the function for a restricted domain of its time arguments and then determine its behavior elsewhere.

The conditions on the function in this domain are most simply expressed in terms of a slightly modified set of Green's functions,

$$\bar{G}_n{}^{\zeta,\,i\tau}(\mathbf{r}_1t_1\cdots\mathbf{r}_nt_n;\,\mathbf{r}_1't_1'\cdots\mathbf{r}_n't_n')$$
$$=e^{i\zeta(t_1\cdots t_n-t_1'\cdots-t_n')}$$
$$\times G_n{}^{i\lambda,\,i\tau}(\mathbf{r}_1t_1\cdots\mathbf{r}_nt_n;\,\mathbf{r}_1't_1'\cdots\mathbf{r}_n't_n'),\quad(5.5)$$

where

$$\zeta=-\lambda/\tau.\quad(5.6)$$

These functions satisfy (5.3) and (5.4) when the differential operator is altered from

$$i(\partial/\partial t_1)+(\nabla_1{}^2/2m)\quad\text{to}\quad i(\partial/\partial t_1)+\zeta+(\nabla_1{}^2/2m).\quad(5.7)$$

As this replacement suggests, the substitution of (5.5) is equivalent to a change in the origin of energy for an N-particle system by $-\zeta N$. This equivalence follows directly from the observation that a change in energy origin induces an additional phase transformation in the expression for the time dependence of the field operator,

$$e^{i(H-\zeta N)(t-t')}\psi(\mathbf{r}t')e^{-i(H-\zeta N)(t-t')}=e^{i\zeta(t-t')}\psi(\mathbf{r}t),$$
$$e^{i(H-\zeta N)(t-t')}\psi^\dagger(\mathbf{r}t')e^{-i(H-\zeta N)(t-t')}=e^{-i\zeta(t-t')}\psi^\dagger(\mathbf{r}t).\quad(5.8)$$

This phase transformation reproduces in the Green's functions the factor introduced in the definition (5.5). Equation (5.8) indicates that $\bar{G}_1{}^{\zeta,\,i\tau}(\mathbf{r}t;\,\mathbf{r}'t')$ may, for $t>t'$, be identified with

$$\bar{G}_{1>}{}^{\zeta,\,i\tau}(\mathbf{r}t;\,\mathbf{r}'t')$$
$$=-i\,\text{Tr}[e^{-i(H-\zeta N)(\tau-t+t')}\psi(\mathbf{r})e^{-i(H-\zeta N)(t-t')}\psi^\dagger(\mathbf{r}')]$$
$$\times[\text{Tr}\,e^{-i(H-\zeta N)\tau}]^{-1},\quad(5.9)$$

while for $t<t'$, it becomes

$$\bar{G}_{1<}{}^{\zeta,\,i\tau}(\mathbf{r}t;\,\mathbf{r}'t')$$
$$=\mp i\,\text{Tr}[e^{-i(H-\zeta N)(\tau-t'+t)}\psi^\dagger(\mathbf{r}')e^{-i(H-\zeta N)(t'-t)}\psi(\mathbf{r})]$$
$$\times[\text{Tr}\,e^{-i(H-\zeta N)\tau}]^{-1}.\quad(5.10)$$

In these expressions, H and ζN occur only in the operator combination $H-\zeta N$. Similarly, the multiparticle Green's functions $\bar{G}_n$ depend only on the operator $H-\zeta N$.

The boundary condition for the Green's functions can be motivated and illustrated with the Green's function, $\bar{G}_1$. From Eqs. (5.9) and (5.10) we see that

$$\bar{G}_{1>}(\mathbf{r}\mathbf{r}';\,t-t'+\tau)=\pm\bar{G}_{1<}(\mathbf{r}\mathbf{r}';\,t-t'),$$

or equivalently

$$\bar{G}_{1>}(t-t')=\pm\bar{G}_{1<}(t-t'-\tau).\quad(5.11)$$

In order that these statements about the two functions $\bar{G}_>$ and $\bar{G}_<$ serve to impose a boundary condition on the single function $\bar{G}_1$, it is necessary that $t-t'<0$ imply $t-t'+\tau>0$ and $t-t'>0$ imply $t-t'-\tau<0$. More concisely, the variable $t-t'$ must satisfy

$$\tau\geqslant|t-t'|.\quad(5.12)$$

This observation suggests the utility of first constructing the function within such a limited domain. The limitation is conveniently enforced by restricting t and t' to the interval $[0,\tau]$. In this interval, the conditions (5.11), applied to $\bar{G}_1$, become the periodicity properties

$$\bar{G}_1(\tau,t')=\pm\bar{G}_1(0,t')\quad\text{and}\quad\bar{G}_1(t,\tau)=\pm\bar{G}_1(t,0).\quad(5.13)$$

The relations (5.11) which imply the boundary condition on $\bar{G}_1$ implicitly require that $\bar{G}_>$ and $\bar{G}_<$, and therefore $\bar{G}$, be defined for suitable complex values of the time difference. The necessity for this complex extension of the Green's functions has its origin in the small imaginary part of τ. The domain of possible extension may be ascertained by examining Eqs. (5.9) and (5.10). The former indicates that $\bar{G}_>$ is defined whenever $\text{Im}\,\tau\leqslant\text{Im}(t-t')\leqslant0$; the latter that $\bar{G}_<$ is defined for $\text{Im}\,\tau\leqslant\text{Im}(t'-t)\leqslant0$. These conditions are summarized in the statement

$$\text{Im}\,\tau\leqslant\text{Im}(t_>-t_<)\leqslant0,\quad(5.14)$$

where the complex numbers $t_>$ and $t_<$ are labelled by the order of their real parts. The restriction (5.14) is analogous to the condition (5.12) we imposed on the real time interval. Both of these conditions are fulfilled by making all times which occur in $\bar{G}$ complex by the

transformation $t \rightarrow t(1-i\epsilon)$ where $-i\epsilon\tau$ is the imaginary part of τ.

Similar considerations apply to the many-particle Green's functions. Thus, the boundary condition on the equation for $\bar{G}_n$ is also determined by the connection between its values for $t_i=0$ and $t_i=\tau$. When $t_i=0$, the operator $\psi(\mathbf{r}_i t_i)$ is the field variable with the earliest time. With the time dependence of (5.8) explicitly introduced, the Green's function takes the form

$$\bar{G}_n(t_i=0) = (-i)^n \epsilon \, \mathrm{Tr}[e^{-i(H-\zeta N)\tau - W} e^{i(H-\zeta N) t_{(1)}}$$
$$\times \chi_{(1)} e^{-i(H-\zeta N)(t_{(1)}-t_{(2)})} \chi_{(2)} \cdots$$
$$\times \chi_{(2n-1)} e^{-i(H-\zeta N) t_{(2n-1)}} \psi(\mathbf{r}_i 0)]. \quad (5.15)$$

In this equation $\chi_{(1)}, \cdots, \chi_{(2n-1)}$, represent the field operators $\psi(\mathbf{r}_j 0)$, $j \neq i$, $j=1, \cdots, n$ and $\psi^\dagger(\mathbf{r}_j'0)$, $j=1, \cdots, n$, labelled from (1) to $(2n-1)$ in the order (from latest to earliest) of the time coordinates, t_j or t_j'. Similarly, when $t_i=\tau$, $\psi(\mathbf{r}_i t_i)$ is the field operator with the latest time, and the Green's function is given by

$$\bar{G}_n(t_i=\tau) = (-i)^n \epsilon \, \mathrm{Tr}[e^{-W} \psi(\mathbf{r}_i 0) e^{-i(H-\zeta N)\tau}$$
$$\times e^{i(H-\zeta N) t_{(1)}} \chi_{(1)} e^{-i(H-\zeta N)(t_{(1)}-t_{(2)})}$$
$$\times \chi_{(2)} \cdots \chi_{(2n-1)} e^{-i(H-\zeta N) t_{(2n-1)}}]. \quad (5.16)$$

In view of the cyclic property of the trace, and the oddness of the permutation that relates the two values of $\bar{G}$, Eqs. (5.15) and (5.16) imply the boundary condition

$$\bar{G}_n(t_i=\tau) = \pm \bar{G}_n(t_i=0), \quad (5.17)$$

with all other time variables arbitrarily fixed. As with the one-particle function, the required complex time extension is achieved by replacing each time variable, including the variable τ, by $t \rightarrow t(1-i\epsilon)$. With this substitution and the condition (5.17) the solutions to Eqs. (5.4) are completely specified.

The boundary conditions may be incorporated directly in the equations by converting the equations to integral form or by restricting the equations to functions which automatically satisfy the boundary conditions. Since the imposed condition is one of periodicity (or antiperiodicity) the second technique, making use of Fourier series, is particularly convenient. All functions with the required periodicity (antiperiodicity) can be constructed from the following complete set of functions labelled by the integer ν, which ranges from $-\infty$ to ∞ :

$$\exp(-i\pi\nu t/\tau). \quad \begin{cases} \nu \text{ even} \quad \text{(B.E.)} \\ \nu \text{ odd} \quad \text{(F.D.)} \end{cases} \quad (5.18)$$

Thus the function $\bar{G}_n$ may be expressed as a multiple Fourier series,

$$\bar{G}_n(\mathbf{r}_1 t_1 \cdots \mathbf{r}_n t_n; \mathbf{r}_1 t_1' \cdots \mathbf{r}_n' t_n')$$
$$= \sum_{\{\nu\}} \exp[-i\pi(\nu_1 t_1 + \cdots + \nu_n t_n - \nu_1' t_1' \cdots - \nu_n' t_n')/\tau]$$
$$\times \tau^{-n} \bar{G}_n(\mathbf{r}_1 \nu_1 \cdots \mathbf{r}_n \nu_n; \mathbf{r}_1' \nu_1' \cdots \mathbf{r}_n' \nu_n'), \quad (5.19)$$

which associates a set of $2n$ frequency indices, $\{\nu\}$, with its $2n$ time coordinates. The Fourier coefficients are determined by the formula

$$\bar{G}_n(\mathbf{r}_1 \nu_1 \cdots \mathbf{r}_n \nu_n; \mathbf{r}_1' \nu_1' \cdots \mathbf{r}_n' \nu_n')$$
$$= \tau^{-n} \int_0^\tau dt_1 \cdots dt_n' \exp[i\pi(\sum \nu t - \sum \nu' t')/\tau]$$
$$\times \bar{G}_n(\mathbf{r}_1 t_1 \cdots \mathbf{r}_n t_n; \mathbf{r}_1' t_1' \cdots \mathbf{r}_n' t_n'). \quad (5.20)$$

The time translational invariance of the functions $\bar{G}(\{t\})$ implies that $\bar{G}(\{\nu\})$ vanishes whenever $\sum \nu \neq \sum \nu'$.

The Fourier transform, $\bar{G}_1(\mathbf{r}\nu; \mathbf{r}'\nu')$, satisfies the equation

$$[(\pi\nu/\tau)+\zeta+(\nabla_1^2/2m)]\bar{G}_1(\mathbf{r}_1\nu_1; \mathbf{r}_1'\nu_1')$$
$$\mp i\tau^{-1} \int d\mathbf{r}_2 \, v(\mathbf{r}_1-\mathbf{r}_2) \sum_{\nu_1''\nu_2\nu_2'} \delta_{\nu_1-\nu_1'', \, \nu_2-\nu_2'}$$
$$\times \bar{G}_2(\mathbf{r}_1\nu_1'' \mathbf{r}_2\nu_2; \mathbf{r}_1'\nu_1' \mathbf{r}_2\nu_2') \exp(i\nu_2'0^+)$$
$$= \delta_{\nu_1\nu_1'}\delta(\mathbf{r}_1-\mathbf{r}_1'), \quad (5.21)$$

and similarly, $\bar{G}_n(\{\nu\})$ obeys

$$[(\pi\nu_1/\tau)+\zeta+(\nabla_1^2/2m)]\bar{G}_n(\mathbf{r}_1\nu_1 \cdots \mathbf{r}_n\nu_n; \mathbf{r}_1'\nu_1' \cdots \mathbf{r}_n'\nu_n')$$
$$\mp i\tau^{-1} \int d\mathbf{r}_{n+1} \, v(\mathbf{r}_1-\mathbf{r}_{n+1})$$
$$\times \sum_{\nu_1''\nu_{n+1}\nu_{n+1}'} \delta_{\nu_1-\nu_1'', \, \nu_{n+1}-\nu_{n+1}'} \exp(i\nu_{n+1}'0^+)$$
$$\times \bar{G}_{n+1}(\mathbf{r}_1\nu_1'' \cdots \mathbf{r}_{n+1}\nu_{n+1}; \mathbf{r}_1'\nu_1' \cdots \mathbf{r}_{n+1}'\nu_{n+1}')$$
$$= \sum_i [\delta_{\nu_1\nu_i'}\delta(\mathbf{r}_1-\mathbf{r}_i')(\pm 1)^{i-1}$$
$$\times \bar{G}_{n-1}(\mathbf{r}_2\nu_2 \cdots \mathbf{r}_n\nu_n; \mathbf{r}_1'\nu_1' \cdots \mathbf{r}_{i-1}'\nu_{i-1}' \mathbf{r}_{i+1}'\nu_{i+1}' \cdots \mathbf{r}_n'\nu_n')].$$
$$(5.22)$$

To illustrate the inversion of these transformed functions and to provide a tool for further manipulations we determine the space-time form of the function

$$\bar{G}_1^0 = [(\pi\nu/\tau)+\zeta+(\nabla^2/2m)]^{-1}\delta(\mathbf{r}-\mathbf{r}').$$

This space-time function, which could be used to convert Eqs. (5.4) to integral equations incorporating the boundary conditions, satisfies the equation

$$[i(\partial/\partial t)-h+\zeta]\bar{G}_1^0(\mathbf{r}t; \mathbf{r}'t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \quad (5.23)$$

and the conditions

$$\bar{G}_1^0(\mathbf{r}\tau; \mathbf{r}'t') = \pm \bar{G}_1^0(\mathbf{r}0; \mathbf{r}'t'), \quad (5.24)$$

where $-\nabla^2/2m$ has been replaced by h. The function $\bar{G}_1^0$ is, therefore, the single-particle Green's function for a system with no interaction. It is first obtained, in a form which holds for all times, by solution of Eqs. (5.23)

and (5.24). In particular, we note that

$$\bar{G}_1{}^0(\mathbf{r}t;\mathbf{r}'t')=\frac{1}{i}\frac{e^{-i(h-\zeta)(t-t')}}{1\mp e^{-i(h-\zeta)\tau}}\delta(\mathbf{r}-\mathbf{r}'), \quad t>t' \qquad (5.26)$$

$$=-\frac{1}{i}\frac{e^{-i(h-\zeta)(t-t')}}{1\mp e^{i(h-\zeta)\tau}}\delta(\mathbf{r}-\mathbf{r}'), \quad t<t' \qquad (5.27)$$

where the denominator is well defined in virtue of the imaginary part of τ. The two expressions (5.26) and (5.27), may be combined as

$$\bar{G}_1{}^0(\mathbf{r}t;\mathbf{r}'t')=\tfrac{1}{2}\{\mp[\cot((h-\zeta)\tau/2)]^{\pm1}-i\epsilon(t-t')\}$$
$$\times e^{-i(h-\zeta)(t-t')}\delta(\mathbf{r}-\mathbf{r}'), \qquad (5.28)$$

where

$$\epsilon(t-t')=\eta_+(t-t')-\eta_-(t-t'). \qquad (5.29)$$

Since these expressions represent the unique solution to (5.23) and (5.24), they must be equivalent to the Fourier series representation,

$$\bar{G}_1{}^0(\mathbf{r}t;\mathbf{r}'t')=\frac{1}{\tau}\sum_\nu\frac{1}{(\pi\nu/\tau)+\zeta-h}e^{-i\pi\nu(t-t')/\tau}$$
$$\times\delta(\mathbf{r}-\mathbf{r}'), \qquad (5.30)$$

for t and t' within $[0,\tau]$. This equivalence may be verified directly by exhibiting an integral whose evaluation by two methods yields the pair of expressions above. The desired integrand must have poles at the points $\pi\nu/\tau$, with residues equal to the terms of the series (5.30). When $t>t'$, a suitable integrand is

$$-[2\pi(\omega+\zeta-h)(1\mp e^{-i\omega\tau})]^{-1}e^{-i\omega(t-t')}\delta(\mathbf{r}-\mathbf{r}'). \qquad (5.31)$$

The integrand vanishes exponentially at ∞ whenever $t>t'$ and $t-t'<\tau$. By choosing a contour which passes between the singularities of $(1\mp e^{-i\omega\tau})^{-1}$ and $(\omega+\zeta-h)^{-1}$, and closing the contour in the lower half plane, we obtain, for $t-t'>0$, the desired summation (5.30). On the other hand, by closing the contour so as to encircle the pole of $(\omega+\zeta-h)^{-1}$, we obtain the expression (5.26) which is therefore equivalent to (5.30) for $0<t-t'<\tau$. A similar procedure reduces the summation (5.30) to (5.27) when $\tau>t'-t>0$.

When there is no interaction we infer that

$$G_1{}^{\alpha\beta}(\mathbf{r}-\mathbf{r}';t-t'),$$

obtained by multiplying $\bar{G}$ by $e^{i\zeta(t-t')}$ and replacing ζ by μ and $i\tau$ by β, is given by

$$G_1{}^{\alpha\beta}(\mathbf{r}t;\mathbf{r}'t')=\frac{1}{i}\frac{e^{-ih(t-t')}}{1\mp e^{-\alpha-\beta h}}\delta(\mathbf{r}-\mathbf{r}'), \quad t>t'$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.32)$$
$$=\pm\frac{1}{i}\frac{e^{-ih(t-t')}}{e^{\alpha+\beta h}\mp1}\delta(\mathbf{r}-\mathbf{r}'), \quad t<t'.$$

Once the functions $\bar{G}_n$ have been determined for all values of the time variables in the interval $[0,\tau]$, they

may be extended to larger values of the time differences. This possibility exists because $\bar{G}_n$ for real t and τ is the boundary value of a function which is analytic for small negative imaginary values of each positive time difference and small positive imaginary values of each negative time difference. From its form in the interval $[0,\tau]$, it may therefore be continued to larger time differences by extension on the appropriate side of each real time-difference axis within the strip

$$0<\mathrm{Im}(t_>-t_<)<\mathrm{Im}\ \tau$$

of analyticity. This continuation determines $\bar{G}_n$, and therefore $G_n{}^{i\lambda,i\tau}$, for complex τ and arbitrary real times. From $\bar{G}_n$, the function $G_n{}^{\alpha\beta}$ is then obtained by letting τ become pure imaginary.

The nature of the continuation may be discussed in more detail when the function, like $\bar{G}_1$, depends only on a single time difference. For such a function we may compare the Fourier expansion (which depends on a single frequency in view of time translational invariance),

$$\bar{G}_1(\mathbf{r}\mathbf{r}';t-t')=\tau^{-1}\sum_\nu e^{i\pi\nu(t-t')/\tau}\bar{G}_1(\mathbf{r}\mathbf{r}';\nu), \quad (5.33)$$

with the generally applicable integral expression (3.32),

$$\bar{G}_1(\mathbf{r}\mathbf{r}';t-t')=\int\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\int\frac{d\omega'}{2\pi}\Big[\mathrm{P}\frac{1}{\omega-\omega'}$$
$$\mp\pi(\cot\tfrac{1}{2}\omega\tau)^{\pm1}\delta(\omega-\omega')\Big]\bar{A}(\mathbf{r}\mathbf{r}'\omega'), \quad (5.34)$$

$$\bar{A}(\mathbf{r}\mathbf{r}'\omega)=A^{-i\zeta\tau,i\tau}(\mathbf{r}\mathbf{r}'\ \omega+\zeta). \qquad (5.35)$$

The ω integration of (5.34) may be performed as in (3.30) and $\bar{G}_1$ written in the form

$$\bar{G}_1(\mathbf{r}\mathbf{r}';t-t')=\int\frac{d\omega}{2\pi}\tfrac{1}{2}[\mp(\cot\tfrac{1}{2}\omega\tau)^{\pm1}-i\epsilon(t-t')]$$
$$\times e^{-i\omega(t-t')}\bar{A}(\mathbf{r}\mathbf{r}'\omega). \qquad (5.36)$$

We now make use of the equivalence of (5.28) and (5.30) to write

$$\bar{G}_1(\mathbf{r}\mathbf{r}';t-t')=\int\frac{d\omega}{2\pi}\sum_\nu\frac{e^{-i\pi\nu(t-t')/\tau}}{\pi\nu-\omega\tau}\bar{A}(\mathbf{r}\mathbf{r}'\omega), \quad (5.37)$$

and thus identify the Fourier coefficient with

$$\bar{G}_1(\mathbf{r}\mathbf{r}';\nu)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\tau}{\pi\nu-\omega\tau}\bar{A}(\mathbf{r}\mathbf{r}'\omega). \qquad (5.38)$$

By comparison of the form of solution obtained for the differential equation (5.21) with (5.38), the function $\bar{A}$ may be inferred and the continuation achieved.

With the exception of the trivial example (5.28), the exact solution of Eqs. (5.4) or (5.22) is presumably not possible. There are, however, a number of approxi-

mation methods which may be employed. In the remainder of this section we shall confine ourselves to particularly simple ones which involve the solution of a subset of Eqs. (5.4) or (5.22) in which the correlation of more than n particles is neglected. The simplest such approximation, and one we would expect to hold when the forces have long range and are slowly varying, is to neglect all dynamical correlations of the particles. In this approximation, which may be called a generalized Hartree approximation, we replace the two-particle correlation function by its form at large space-time separations, a product of single-particle functions. Since the two-particle function must be symmetric (or antisymmetric), it is natural to make this replacement in such a way that the symmetry is preserved. We therefore set

$$\bar{G}_2(\mathbf{r}_1 t_1 \mathbf{r}_2 t_2; \mathbf{r}_1' t_1' \mathbf{r}_2' t_2') \cong \bar{G}_1(\mathbf{r}_1 t_1; \mathbf{r}_1' t_1') \bar{G}_1(\mathbf{r}_2 t_2; \mathbf{r}_2' t_2')$$
$$\pm \bar{G}_1(\mathbf{r}_1 t_1; \mathbf{r}_2' t_2') \bar{G}_1(\mathbf{r}_2 t_2; \mathbf{r}_1' t_1'), \quad (5.39)$$

which is, incidentally, always correct for noninteracting particles. We are then led to the equation

$$[i(\partial/\partial t_1) + (\mathbf{\nabla}_1^2/2m) + \zeta]\bar{G}_1(\mathbf{r}_1 t_1; \mathbf{r}_1' t_1')$$

$$-\int v(\mathbf{r}_1 - \mathbf{r}_2) d\mathbf{r}_2 \langle n(\mathbf{r}_2) \rangle^{i\lambda, i\tau} \bar{G}_1(\mathbf{r}_1 t_1; \mathbf{r}_1' t_1')$$

$$-i\int v(\mathbf{r}_1 - \mathbf{r}_2) d\mathbf{r}_2 \ \bar{G}_1(\mathbf{r}_1 t_1; \mathbf{r}_2 t_1^+) \bar{G}_1(\mathbf{r}_2 t_1; \mathbf{r}_1' t_1')$$
$$= \delta(\mathbf{r}_1 - \mathbf{1}_1')\delta(t_1 - t_1'), \quad (5.40)$$

where $\langle n(\mathbf{r}) \rangle$ is a local particle density,

$$\pm iG_1^{i\lambda, i\tau}(\mathbf{r}t; \mathbf{r}t^+) = \pm i\bar{G}_1(\mathbf{r}t; \mathbf{r}t^+).$$

The interaction terms in this expression may be viewed as an effective potential which is nonlocal but time independent. The corresponding equation for the Fourier coefficient Green's function (5.21) may be reduced to

$$[(\pi\nu/\tau) + (\mathbf{\nabla}_1^2/2m) + \zeta]\bar{G}_1(\mathbf{r}_1\nu; \mathbf{r}_1'\nu)$$

$$-\int v(\mathbf{r}_1 - \mathbf{r}_2) d\mathbf{r}_2 \langle n(\mathbf{r}_2) \rangle^{i\lambda, i\tau} \bar{G}_1(\mathbf{r}_1\nu; \mathbf{r}_1'\nu)$$

$$-i\tau^{-1}\sum_{\nu'}\int v(\mathbf{r}_1 - \mathbf{r}_2) d\mathbf{r}_2 \ \bar{G}_1(\mathbf{r}_1\nu'; \mathbf{r}_2\nu')$$

$$\times \bar{G}_1(\mathbf{r}_2\nu; \mathbf{r}_1'\nu)e^{i\nu'0+} = \delta(\mathbf{r}_1 - \mathbf{r}_1'). \quad (5.41)$$

The equation is of the form (5.23), where h is the matrix

$$\langle \mathbf{r}|h|\mathbf{r}' \rangle$$

$$= \left[-(\mathbf{\nabla}^2/2m) + \int v(\mathbf{r} - \mathbf{r}'') \langle n(\mathbf{r}'') \rangle^{\zeta, i\tau} d\mathbf{r}'' \right]\delta(\mathbf{r} - \mathbf{r}')$$

$$+ i\sum_{\nu} \tau^{-1} v(\mathbf{r} - \mathbf{r}') \bar{G}_1(\mathbf{r}\nu; \mathbf{r}'\nu)e^{i\nu0+}. \quad (5.42)$$

The Green's function therefore satisfies Eqs. (5.26) and (5.27) which, in this case, define an equation for $\sum_{\nu} \bar{G}_1(\mathbf{r}\nu; \mathbf{r}'\nu)e^{i\nu0+}$ from which $\bar{G}_1$ is determined.

In a uniform system, with

$$\bar{G}_1(\mathbf{r}\mathbf{r}'; t - t') = \bar{G}_1(\mathbf{r} - \mathbf{r}'; t - t'),$$

the approximation (5.40) is conveniently discussed in momentum space, where

$$\bar{G}_1(\mathbf{r} - \mathbf{r}'; t - t') = \int \frac{d\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{r} - \mathbf{r}')} \bar{G}_1(\mathbf{p}; t - t'). \quad (5.43)$$

The matrix (5.42) is then diagonal and its elements satisfy

$$h(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + \int v(\mathbf{r}'') d\mathbf{r}'' \langle n \rangle$$
$$\pm \int v(\mathbf{p} - \mathbf{p}') \langle n(\mathbf{p}') \rangle^{\zeta, i\tau} \frac{d\mathbf{p}'}{(2\pi)^3}. \quad (5.44)$$

The function $v(\mathbf{p})$ occurring in (5.44) is defined by

$$v(\mathbf{p}) = \int e^{-i\mathbf{p}\cdot\mathbf{r}} v(\mathbf{r}) d\mathbf{r}, \quad (5.45)$$

and $\langle n(\mathbf{p}) \rangle^{\zeta, i\tau}$ by

$$\langle n(\mathbf{p}) \rangle^{\zeta, i\tau} = [e^{i(h-\zeta)\tau} \mp 1]^{-1}. \quad (5.46)$$

Equations (5.26) and (5.27) then define an integral equation determining $h(\mathbf{p})$ and $n(\mathbf{p})$ which may be cast in the perspicuous form

$$\langle n(\mathbf{p}) \rangle^{\alpha\beta} = \left\{ \exp\left[\beta\left(\frac{\mathbf{p}^2}{2m} + nv(\mathbf{p}=0) \right.\right.\right.$$
$$\left.\left.\left. \pm \int v(\mathbf{p} - \mathbf{p}') \langle n(\mathbf{p}') \rangle^{\alpha\beta} \frac{d\mathbf{p}'}{(2\pi)^3} \right) + \alpha \right] \mp 1 \right\}^{-1}, \quad (5.47)$$

by transforming to $i\tau = \beta$, $\zeta = -\alpha/\beta$ and setting the particle density equal to n. Equation (5.47) is a special case of (3.68) wherein the function

$$A(\mathbf{p}\omega) = 2\pi\delta(\omega - h(\mathbf{p}))$$

associates a definite frequency with each momentum. The last nonlocal term in the matrix h is an exchange term which produces a velocity dependence in the "effective single-particle potential." For a long-range force this exchange term is generally negligible compared with the preceding direct interaction term; for a short-range force they are approximately equal.

Although this association of a definite frequency with each momentum is specific to the Hartree approxima-

tion, the integral statement

$$\int \frac{d\omega}{2\pi} \omega A(\mathbf{p}\omega)$$

$$= \int \frac{d\omega}{2\pi} \frac{\mathbf{p}^2}{2m} A(\mathbf{p}\omega) + v(0) \int \frac{d\omega}{2\pi} \frac{d\mathbf{p}'}{(2\pi)^3} A(\mathbf{p}'\omega) \frac{1}{e^{\alpha+\beta\omega} \mp 1}$$

$$\pm \int \frac{d\omega}{2\pi} \frac{d\mathbf{p}'}{(2\pi)^3} v(\mathbf{p}-\mathbf{p}') A(\mathbf{p}'\omega) \frac{1}{e^{\alpha+\beta\omega} \mp 1} \quad (5.48)$$

is more generally valid. The derivation of this result exemplifies a class of relations based upon the high-frequency behavior of the Green's function equations. For any potential which is bounded, and whose effect therefore vanishes at high frequencies, we may conclude from Eq. (5.22) that $\bar{G}_n(\{\nu\})$ behaves as ν^{-1} as any argument ν approaches infinity. Indeed the expression

$$\lim_{\nu \to \infty} \pi \nu \bar{G}_1(\mathbf{r}\nu; \mathbf{r}'\nu) = \delta(\mathbf{r}-\mathbf{r}')\tau \quad (5.49)$$

may be used with (5.38) to rederive

$$\int \frac{d\omega}{2\pi} \bar{A}(\mathbf{r}\mathbf{r}'\omega) = \delta(\mathbf{r}-\mathbf{r}'). \quad (5.50)$$

Additional relations are obtained by keeping subsequent terms in the high-frequency expansion. Thus we determine a second identity by operating on (5.21) from the right with $[\pi\nu - (h_1' - \zeta)\tau]$:

$$\{[\pi\nu_1 - (h_1-\zeta)\tau]\bar{G}_1(\mathbf{r}_1\nu_1; \mathbf{r}_1'\nu_1) - \tau\delta(\mathbf{r}_1-\mathbf{r}_1')\}$$

$$\times [\pi\nu_1 - (h_1'-\zeta)\tau]$$

$$= \pm i \sum_{\nu_2} \tau \int v(\mathbf{r}_1-\mathbf{r}_2) d\mathbf{r}_2 [\bar{G}_1(\mathbf{r}_2\nu_2; \mathbf{r}_2\nu_2)\delta(\mathbf{r}_1-\mathbf{r}_1')$$

$$\pm \bar{G}_1(\mathbf{r}_1\nu_2; \mathbf{r}_2\nu_2)\delta(\mathbf{r}_2-\mathbf{r}_1')]$$

$$- \sum_{\nu_2,\nu_3} \int v(\mathbf{r}_1-\mathbf{r}_2) d\mathbf{r}_2$$

$$\times \bar{G}_3(\mathbf{r}_1\nu_1\mathbf{r}_2\nu_2\mathbf{r}_3\nu_3; \mathbf{r}_1'\nu_1\mathbf{r}_2\nu_2\mathbf{r}_3\nu_3) d\mathbf{r}_3 v(\mathbf{r}_3-\mathbf{r}_1') \quad (5.51)$$

and, as in (5.26), letting $h = -\nabla^2/2m$.

For a bounded potential, this expression explicitly exhibits the ν^{-1} and ν^{-2} terms in the asymptotic expansion of the Green's function. Comparison of (5.51) with (5.38) then yields the relation

$$\int \frac{d\omega}{2\pi} (\omega - h) A(\mathbf{r}\mathbf{r}'\omega)$$

$$= \int \frac{d\omega}{2\pi} v(\mathbf{r}-\mathbf{r}'') d\mathbf{r}'' A(\mathbf{r}''\mathbf{r}''\omega) \frac{1}{e^{\alpha+\beta\omega} \mp 1} \delta(\mathbf{r}-\mathbf{r}')$$

$$\pm \int \frac{d\omega}{2\pi} v(\mathbf{r}-\mathbf{r}') A(\mathbf{r}\mathbf{r}'\omega) \frac{1}{e^{\alpha+\beta\omega} \mp 1}, \quad (5.52)$$

which, in a homogeneous medium, reduces to the form (5.48).

A second approximation, in the chain in which Eq. (5.40) is first, is obtained by considering only two-particle correlations. In addition to improving the treatment of those systems for which the generalized Hartree approximation is a satisfactory starting point, this extension should apply to systems in which strong correlations do occur but at distances small compared to the mean interparticle spacing, where two-particle correlations should predominate. Since the inhomogeneous term and boundary conditions of the differential equation for the two-particle correlation function are the same as those of the symmetrized product of single-particle correlation functions, we write the equation determining the measure of correlation as

$$[i(\partial/\partial t_1) - h_1 + \zeta]\{\bar{G}_2(12; 1'2')$$

$$- [\bar{G}_1(1; 1')\bar{G}_1(2; 2') \pm \bar{G}_1(1; 2')\bar{G}_1(2; 1')]\}$$

$$\mp i \int v(1,3)\{\bar{G}_3(123; 1'2'3) - [\bar{G}_2(13; 1'3)\bar{G}_1(2; 2')$$

$$\pm \bar{G}_2(13; 2'3)\bar{G}_1(2; 1')]\} = 0. \quad (5.53)$$

In this equation space-time indices have been suppressed, $v(1,3)$ represents $v(\mathbf{r}_1-\mathbf{r}_3)\delta(t_1-t_3)$, and integration extends to all additional space-time variables under the sign of integration. Equation (5.53) admits an approximation one step beyond (5.39). We recognize that the right-hand side involves the difference between a three-particle Green's function in the space-time region where two particles interact, and the product of Green's functions which take into account the correlation between this pair neglecting their correlation with a third. For long-ranged slowly varying forces and for short-ranged forces at low densities we would expect that G_3 might be replaced by a sum of functions of the form $G_2 G_1$, which agrees with G_3 at large separations and also approximately within the range of the interacting pair. The function with this desired behavior

$$\bar{G}_3(123; 1'2'3)$$

$$\cong \bar{G}_2(13; 1'3)\bar{G}_1(2; 2') + \bar{G}_2(13; 32')\bar{G}_1(2; 1')$$

$$+ \bar{G}_2(13; 2'1')\bar{G}_1(2; 3), \quad (5.54)$$

leads to an approximate form for (5.53),

$$[i(\partial/\partial t_1) - h_1 + \zeta]\{\bar{G}_2(12; 1'2')$$

$$- [\bar{G}_1(1; 1')\bar{G}_1(2; 2') \pm \bar{G}_1(1; 2')\bar{G}_1(2; 1')]\}$$

$$- i \int v(1,3)\bar{G}_2(13; 1'2')\bar{G}_1(2; 3) = 0. \quad (5.55)$$

In the remainder of the section we shall make the additional replacement of $\bar{G}_1(2; 3)$ by $\bar{G}_1^0(2; 3)$ in (5.55). This replacement will be a permissible first approximation in the determination of the combination $v(1,2) \times \bar{G}_2(12; 1'2')$ that occurs in the one-particle equation

whenever the interaction of a pair of particles with others may be neglected while their mutual interaction is producing correlations. The error entailed in the pair of approximations is given by the right-hand side of the rigorous equation,

$$[i(\partial/\partial t_1)-h_1+\zeta][i(\partial/\partial t_2)-h_2+\zeta]\{\bar{G}_2(12;1'2')$$

$$-[\bar{G}_1(1;1')\bar{G}_1(2;2')\pm\bar{G}_1(1;2')\bar{G}_1(2;1')]\}$$

$$-iv(1,2)\bar{G}_2(12;1'2')$$

$$=-\int v(1,3)v(2,4)\{\bar{G}_4(1234;1'2'34)$$

$$-[\bar{G}_2(13;1'3)\bar{G}_2(24;2'4)$$

$$\pm\bar{G}_2(13;2'3)\bar{G}_2(24;1'4)]\}, \quad (5.56)$$

obtained by applying $[i(\partial/\partial t_2)-h_2+\zeta]$ to (5.53). Much of the subsequent analysis of the simplified equation obtained by neglecting this remainder carries over to more accurate equations derived in Sec. VI.

It is convenient to introduce a matrix notation in which multiplication involves integrating the coordinate matrix indices over all space and over the time interval $[0,\tau]$, the potential matrix is given by

$$\langle 12|v_{12}|1'2'\rangle=v(1,2)\delta(1,1')\delta(2,2'), \quad (5.57)$$

and the appropriately symmetrized unit matrix is represented by

$$1_{12}=\delta(1,1')\delta(2,2')\pm\delta(1,2')\delta(2,1'). \quad (5.58)$$

The approximation to (5.53) or (5.56) is then succinctly expressed as

$$\bar{G}_{12}=\bar{G}_1\bar{G}_2 1_{12}+i\bar{G}_1^0\bar{G}_2^0 v_{12}\bar{G}_{12}, \quad (5.59)$$

with the subscripts on the Green's functions denoting particle coordinates. A convenient auxiliary for discussing this equation is the matrix Ω_{12} which satisfies

$$\Omega_{12}=1_{12}+i\bar{G}_1^0\bar{G}_2^0 v_{12}\Omega_{12}$$

$$=1_{12}+[(\bar{G}_1^0)^{-1}(\bar{G}_2^0)^{-1}-iv_{12}]^{-1}iv_{12}1_{12}, \quad (5.60)$$

$$\bar{G}_{12}=\Omega_{12}\bar{G}_1\bar{G}_2.$$

By employing the relation,

$$\bar{G}_1^0\bar{G}_2^0=[\bar{G}_1^0+\bar{G}_2^0][(\bar{G}_1^0)^{-1}+(\bar{G}_2^0)^{-1}]^{-1}, \quad (5.61)$$

we may rewrite (5.60) in the form

$$\Omega_{12}-i[\bar{G}_1^0+\bar{G}_2^0][(\bar{G}_1^0)^{-1}+(\bar{G}_2^0)^{-1}]^{-1}v_{12}\Omega_{12}=1_{12}. \quad (5.62)$$

The matrix Ω occurs in the $\bar{G}_1$ equations (5.3) and (5.21) in the form $v\Omega$, where v is an instantaneous interaction. Consequently, for determining $\bar{G}_1$, it is sufficient to find the submatrix in which the time variables of the two left indices of Ω are set equal. Since the second term contains, as we read from right to left, Ω with equal left time indices, the operator $(\bar{G}_1^0)^{-1}+(\bar{G}_2^0)^{-1}$ which translates the total time but does not alter relative times, and the combination $\bar{G}_1^0+\bar{G}_2^0$, which, on the one hand, changes either of the time arguments but not

both, and, on the other hand, leaves the pair of time arguments equal, we may conclude that it is only the combination $\bar{G}_1^0+\bar{G}_2^0$, at vanishing time difference, which occurs. This combination has a well-defined limit as the time difference approaches zero (reflecting the continuity of the Green's function as the difference in time indices of two ψ or two $\psi^\dagger$ field variables passes through zero). The equation for this submatrix may be written explicitly in the form

$$\langle r_1 r_2 t|\Omega|r_1'r_2't'\rangle\pm\tfrac{1}{2}i\{[\cot(\tfrac{1}{2}(h_1-\zeta)\tau)]^{\pm1}$$

$$+[\cot(\tfrac{1}{2}(h_2-\zeta)\tau)]^{\pm1}\}\int_0^\tau d\bar{t}\int dr_1''\int dr_2''$$

$$\times\langle r_1 r_2 t|[i(\partial/\partial t)+2\zeta-h_1-h_2]^{-1}|r_1''r_2''\bar{t}\rangle$$

$$\times v(r_1''-r_2'')\langle r_1''r_2''\bar{t}|\Omega|r_1'r_2't'\rangle$$

$$=[\delta(r_1-r_1')\delta(r_2-r_2')$$

$$\pm\delta(r_1-r_2')\delta(r_2-r_1')]\delta(t-t'), \quad (5.63)$$

by removing a δ-function from the equation for

$$\langle r_1 r_2 t|\Omega|r_1't_1'r_2't_2'\rangle=\langle r_1 r_2 t|\Omega|r_1'r_2't_1'\rangle\delta(t_1'-t_2').$$

Since the single-time functions refer to a pair of equal times, the differential operator and the matrix Ω satisfy periodic boundary conditions in the interval $[0,\tau]$ for fermions as well as bosons. This property is automatically built into the equation by understanding that $i\partial/\partial t$ is represented by $\pi(\nu_1+\nu_2)/\tau=\pi\nu/\tau$, and that ν must be even. In terms of this frequency $\pi\nu/\tau$, Eq. (5.63) becomes

$$[(\pi\nu/\tau)+2\zeta-h_1-h_2]\langle r_1 r_2|\Omega(\nu)|r_1'r_2'\rangle$$

$$-\tfrac{1}{2}\{[\coth(\tfrac{1}{2}i(h_1-\zeta)\tau)]^{\pm1}+[\coth(\tfrac{1}{2}i(h_2-\zeta)\tau)]^{\pm1}\}$$

$$\times v(r_1-r_2)\langle r_1 r_2|\Omega(\nu)|r_1'r_2'\rangle$$

$$=[(\pi\nu/\tau)+2\zeta-h_1-h_2][\delta(r_1-r_1')\delta(r_2-r_2')$$

$$\pm\delta(r_1-r_2')\delta(r_2-r_1')]. \quad (5.64)$$

Because of the spatial translational invariance of Ω, and its simple dependence on the center-of-mass coordinate $\tfrac{1}{2}(r_1+r_2)$, this equation is best discussed in terms of the variables

$$R=\tfrac{1}{2}(r_1+r_2), \quad r=r_1-r_2, \quad (5.65)$$

and the operators

$$P=(1/i)(\nabla_1+\nabla_2), \quad p=(1/2i)(\nabla_1-\nabla_2). \quad (5.66)$$

Indeed the essential problem is construction of $\mathcal{G}(rr';P,\nu)$ which satisfies

$$\left(\frac{\pi\nu}{\tau}+2\zeta-\frac{P^2}{4m}+\frac{\nabla^2}{m}\right)\mathcal{G}(r,r';P,\nu)$$

$$-\tfrac{1}{2}[\{\coth[((\tfrac{1}{2}P+p)^2/2m-\zeta)\tfrac{1}{2}i\tau]\}^{\pm1}$$

$$+\{\coth[((\tfrac{1}{2}P-p)^2/2m-\zeta)\tfrac{1}{2}i\tau]\}^{\pm1}]$$

$$\times v(r)\mathcal{G}(rr';P,\nu)=\delta(r-r'), \quad (5.67)$$

in the momentum-space representation for the total momentum $\mathbf{P}$. In terms of $\mathcal{G}$, Ω is given by

$$\langle \mathbf{r}_1 \mathbf{r}_2 | \Omega(\nu) | \mathbf{r}_1' \mathbf{r}_2' \rangle$$

$$= \int \frac{d\mathbf{P}}{(2\pi)^3} e^{i\mathbf{P}\cdot(\mathbf{R}-\mathbf{R}')} [\mathcal{G}(\mathbf{r}\mathbf{r}'; \mathbf{P}\nu) \pm \mathcal{G}(\mathbf{r}, -\mathbf{r}'; \mathbf{P}\nu)]$$

$$\times \left(\frac{\pi\nu}{\tau} + 2\zeta - \frac{\mathbf{P}^2}{4m} + \frac{\nabla'^2}{m} \right), \quad (5.68)$$

where the differential operator acts on the primed variable from the right.

The function $\mathcal{G}$ is, of course, also a Green's function. Indeed, it is very similar to the Green's function ordinarily encountered in the two-particle problem. More specifically, if we make the replacements $E = (\pi\nu/\tau) - (P^2/4m)$, $i\tau = \beta$, $\zeta = \mu$, and $\mu = -\alpha/\beta$ we recognize $\mathcal{G}$ as the inverse of

$$E + 2\mu - (\mathbf{P}^2/m) - \tfrac{1}{2}[\{\coth\tfrac{1}{2}[\beta(\tfrac{1}{2}\mathbf{P}-\mathbf{p})^2/2m + \alpha]\}^{\pm 1}$$
$$+ \{\coth\tfrac{1}{2}[\beta(\tfrac{1}{2}\mathbf{P}+\mathbf{p})^2/2m + \alpha]\}^{\pm 1}]v. \quad (5.69)$$

In this form physical interpretation is possible. As α approaches ∞ (which corresponds to vanishing density), the operator reduces to the one occurring in the Schrödinger equation for the relative coordinate of a pair of particles. With μ fixed, the equation describes interaction in the presence of a medium which results in an effective potential whose value depends qualitatively on the relation of energies of the individual particles to the chemical potential, μ.

Following conventional scattering theory, we introduce the useful auxiliaries

$$\hat{t}(\mathbf{P},\nu)\mathcal{G}^{(0)}(\mathbf{P},\nu) = v\mathcal{G}(\mathbf{P},\nu),$$
$$\mathcal{G}^{(0)}(\mathbf{P},\nu) = [(\pi\nu/\tau) + 2\zeta - h_1 - h_2]^{-1}, \quad (5.70)$$

in a matrix notation in the relative coordinate spatial variable. In virtue of (5.68), the matrix $\hat{t}$ satisfies

$$\langle \mathbf{r} | \hat{t}(\mathbf{P},\nu) | \mathbf{r}' \rangle \pm \langle \mathbf{r} | \hat{t}(\mathbf{P},\nu) | -\mathbf{r}' \rangle$$

$$= v(\mathbf{r}) \int \langle \mathbf{r}_1 \mathbf{r}_2 | \Omega(\nu) | \mathbf{r}_1' \mathbf{r}_2' \rangle d(\mathbf{R}-\mathbf{R}') e^{-i\mathbf{P}\cdot(\mathbf{R}-\mathbf{R}')}. \quad (5.71)$$

We may also write $\hat{t}$ as

$$\hat{t} = v + v\mathcal{G}\{[\coth(\tfrac{1}{2}(h_1-\zeta)i\tau)]^{\pm 1}$$
$$+ [\coth(\tfrac{1}{2}(h_2-\zeta)i\tau)]^{\pm 1}\}v, \quad (5.72)$$

or in terms of the equation

$$\hat{t} = v + i\mathcal{G}^{(0)}\tfrac{1}{2}\{[\coth(\tfrac{1}{2}(h_1-\zeta)i\tau)]^{\pm 1}$$
$$+ [\coth(\tfrac{1}{2}(h_2-\zeta)i\tau)]^{\pm 1}\}v, \quad (5.73)$$

which is exactly soluble for several interesting problems. Equation (5.73) also provides a basis for various approximation techniques. For example, the solution to the ordinary Schrödinger equation for any potential

determines $\hat{t}^{(0)}$ such that

$$\hat{t}^{(0)} = v + v\mathcal{G}^{(0)}\hat{t}^{(0)}. \quad (5.74)$$

Together with (5.73), this equation implies that

$$\hat{t} = \hat{t}^{(0)} \pm i\mathcal{G}^{(0)}\{[\exp(i(h_1-\zeta)\tau) \mp 1]^{-1}$$
$$+ [\exp(i(h_2-\zeta)\tau) \mp 1]^{-1}\}\hat{t}^{(0)}. \quad (5.75)$$

The latter equation, from which the potential has been removed, permits an iterative solution when there are strong repulsive short-range forces. Indeed in some circumstances, it is possible to solve (5.75) approximately by other than iterative means, and also to set up an effective-range theory for operator $\hat{t}$.

As we have indicated, a major reason for determining $\hat{t}$ is to obtain an approximate equation for the single-particle Green's function which takes certain two-particle correlations into account. Such an equation now follows from (5.21), (5.60), and (5.71), in the form

$$\left(\frac{\pi\nu}{\tau} + \zeta - h_1 \right) \bar{G}_1(\mathbf{r}_1\nu; \mathbf{r}_1'\nu)$$

$$\mp \frac{i}{\tau} \sum_{\nu'} \int [\langle \mathbf{r}_1 - \mathbf{r}_2 | (\hat{t} \pm \hat{t}')(\mathbf{P}, \nu+\nu') | \mathbf{r}_1'' - \mathbf{r}_2'' \rangle]$$

$$\times e^{i\mathbf{P}\cdot(\mathbf{R}-\mathbf{R}'')} \frac{d\mathbf{P}}{(2\pi)^3} d\mathbf{r}_1'' d\mathbf{r}_2'' d\mathbf{r}_2 \bar{G}_1(\mathbf{r}_2''\nu'; \mathbf{r}_2\nu')$$

$$\times \bar{G}_1(\mathbf{r}_1''\nu; \mathbf{r}_1'\nu) = \delta(\mathbf{r}_1 - \mathbf{r}_1'); \quad (5.76)$$

where

$$\langle \mathbf{r} | \hat{t}' | \mathbf{r}' \rangle = \langle \mathbf{r} | \hat{t} | -\mathbf{r}' \rangle.$$

The effective one-particle potential in this equation is frequency or time dependent, and consequently the one-particle Green's function can no longer be described, as in the Hartree approximation, by an effective single-particle Hamiltonian.

As in the Hartree approximation, this equation is particularly tractable for systems which may be treated as uniform, that is, for which the Green's function is given by

$$\bar{G}_1(\mathbf{r}-\mathbf{r}'; t-t') = \int \frac{d\mathbf{p}}{(2\pi)^3} \tau^{-1} \sum_{\nu} \bar{G}_1(\mathbf{p},\nu)$$

$$\times e^{-i\pi\nu(t-t')/\tau} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}. \quad (5.77)$$

For such systems, Eq. (5.76) reduces to

$$\bar{G}_1(\mathbf{p},\nu) = \left[\frac{\pi\nu}{\tau} + \zeta - \frac{\mathbf{p}^2}{2m} \mp \frac{i}{\tau} \sum_{\nu'} \int \frac{d\mathbf{p}'}{(2\pi)^3} \right.$$

$$\times \langle \tfrac{1}{2}(\mathbf{p}-\mathbf{p}') | (\hat{t} \pm \hat{t}')(\mathbf{p}+\mathbf{p}', \nu+\nu') | \tfrac{1}{2}(\mathbf{p}-\mathbf{p}') \rangle$$

$$\left. \times \bar{G}_1(\mathbf{p}'\nu') \exp(i\nu'0^+) \right]^{-1}, \quad (5.78)$$

where

$$\langle \mathbf{p} | \hat{t}(\mathbf{P},\nu) | \mathbf{q} \rangle = \int e^{-i\mathbf{p}\cdot\mathbf{r}} \langle \mathbf{r} | \hat{t}(\mathbf{P},\nu) | \mathbf{r}' \rangle e^{i\mathbf{q}\cdot\mathbf{r}'} d\mathbf{r} d\mathbf{r}'. \quad (5.79)$$

By employing Eq. (5.38), we conclude

$$\tau \int \frac{d\omega'}{2\pi} \frac{\bar{A}(\mathbf{p'}\omega')}{\pi\nu - \omega'\tau}$$

$$= \left[\frac{\pi\nu}{\tau} + \zeta - \frac{\mathbf{p}^2}{2m} \mp \frac{i}{\tau} \sum_{\nu'} \int \frac{d\mathbf{p'}}{(2\pi)^3} \int \frac{d\omega'}{2\pi} \right.$$

$$\times \langle \tfrac{1}{2}(\mathbf{p}-\mathbf{p'})|(\bar{t}\pm\bar{t}')(\mathbf{p}+\mathbf{p'}, \nu+\nu')|\tfrac{1}{2}(\mathbf{p}-\mathbf{p'})\rangle$$

$$\times \left(\frac{\pi\nu'}{\tau} - \omega' \right)^{-1} \bar{A}(\mathbf{p'}\omega') \exp(i\nu'0^+) \right]^{-1}. \quad (5.80)$$

In order to proceed with the analysis of this equation we suppose that the eigenfunction expansion of $\mathcal{G}$ is known. In the α, β formulation as $\alpha \to \infty$, this expansion would be the familiar one for $(E-H_{12})^{-1}$ where H_{12} is the Hermitian operator that characterizes two-particle interactions. The existence of such a representation more generally is inferred by treating any potential as the limit of a sum of factorizable potentials for which its existence can be explicitly demonstrated. From the expansion and Eq. (5.72) a representation for $\bar{t}(\mathbf{P},\nu)$ may be obtained:

$$\langle \mathbf{r}|\bar{t}(\mathbf{P}, \nu+\nu')|\mathbf{r'}\rangle = v(\mathbf{r})\delta(\mathbf{r}-\mathbf{r'})$$

$$+ \tau \sum_n \frac{v(\mathbf{r})\rho_n(\mathbf{rr'})v(\mathbf{r'})}{\pi(\nu+\nu') - \bar{E}_n\tau}, \quad (5.81)$$

in terms of coefficients, $\rho_n(\mathbf{rr'})$, and poles, $\bar{E}_n$, which, in general, need not be real. With the aid of this expression the summation that occurs in (5.8) may be evaluated. We obtain for the inverse of $\bar{G}_1$,

$$[\bar{G}_1(\mathbf{p}\nu)]^{-1} = \left[\frac{\pi\nu}{\tau} + \zeta - \frac{\mathbf{p}^2}{2m} - \bar{V}\left(\mathbf{p}, \frac{\pi\nu}{\tau}\right) \right], \quad (5.82)$$

where

$$\bar{V}\left(\mathbf{p}, \frac{\pi\nu}{\tau}\right) = \int \frac{d\mathbf{p'}}{(2\pi)^3} n(\mathbf{p'})[v(\mathbf{p}=0)\pm v(\mathbf{p}-\mathbf{p'})]$$

$$+ \int \frac{d\mathbf{p'}}{(2\pi)^3} \int \frac{d\omega'}{2\pi} \sum_n \frac{\bar{r}_n(\mathbf{p},\mathbf{p'})}{(\pi\nu/\tau) + \omega' - \bar{E}_n}$$

$$\times \left(\frac{1}{[\exp(i\omega'\tau) \mp 1]} \mp \frac{1}{[\exp(i\bar{E}_n\tau) - 1]} \right)$$

$$\times \bar{A}(\mathbf{p'}\omega'), \quad (5.83)$$

and

$$\bar{r}_n(\mathbf{p},\mathbf{p'}) = \int e^{-i\frac{1}{2}(\mathbf{p}-\mathbf{p'})\cdot\mathbf{r}} v(\mathbf{r})\rho_n(\mathbf{rr'})v(\mathbf{r'})$$

$$\times [e^{i\frac{1}{2}(\mathbf{p}-\mathbf{p'})\cdot\mathbf{r'}} \pm e^{-i\frac{1}{2}(\mathbf{p}-\mathbf{p'})\cdot\mathbf{r'}}]. \quad (5.84)$$

We may now reduce (5.80) to an integral equation for $\bar{A}(\mathbf{p}\omega)$. This reduction is immediately achieved by observing that both the left- and right-hand sides of (5.80) define analytic functions when $\pi\nu/\tau$ is replaced

by the general complex variable $z=\omega+i\delta$. Since these functions agree on a set of points which have a limit point at ∞, and the functions are analytic at ∞, they are equal everywhere and

$$\bar{A}(\mathbf{p}\omega) = \lim_{\delta\to 0} \frac{1}{2\pi i} \int d\omega' \left[\frac{1}{\omega-\omega'-i\delta} - \frac{1}{\omega-\omega'+i\delta} \right] \bar{A}(\mathbf{p}\omega') \quad (5.85)$$

is given by

$$\bar{A}(\mathbf{p}\omega) = \frac{\Gamma(\mathbf{p}\omega)}{[\omega-\epsilon(\mathbf{p}\omega)]^2 + [\frac{1}{2}\Gamma(\mathbf{p}\omega)]^2}, \quad (5.86)$$

where

$$\epsilon(\mathbf{p}\omega) - \frac{\mathbf{p}^2}{2m} + \zeta = \lim_{\delta\to 0} \tfrac{1}{2}[\bar{V}(\mathbf{p}, \omega+i\delta) + \bar{V}(\mathbf{p}, \omega-i\delta)] \quad (5.87)$$

$$= \int \frac{d\mathbf{p'}}{(2\pi)^3} n(\mathbf{p'})[v(\mathbf{p}=0)\pm v(\mathbf{p}-\mathbf{p'})]$$

$$+ \int \frac{d\mathbf{p'}}{(2\pi)^3} P \int \frac{d\omega'}{2\pi} \sum_n \frac{\bar{r}_n(\mathbf{pp'})}{\omega+\omega' - \bar{E}_n}$$

$$\times \{[\exp(i\omega'\tau) \mp 1]^{-1}$$

$$\mp [\exp(i\bar{E}_n\tau) - 1]^{-1}\}\bar{A}(\mathbf{p'}\omega'), \quad (5.88)$$

and

$$\Gamma(\mathbf{p}\omega) = \lim_{\delta\to 0} \frac{1}{i}[\bar{V}(\mathbf{p}, \omega-i\delta) - \bar{V}(\mathbf{p}, \omega+i\delta) + 2i\delta] \quad (5.89)$$

$$= \lim_{\delta\to 0} 2\delta \left[1 + \int \frac{d\mathbf{p'}}{(2\pi)^3} \int \frac{d\omega'}{2\pi} \right.$$

$$\times \sum_n \frac{\bar{r}_n(\mathbf{pp'})}{(\omega+\omega'-\bar{E}_n)^2 + \delta^2}\{[\exp(i\omega'\tau) \mp 1]^{-1}$$

$$\mp [\exp(i\bar{E}_n\tau) - 1]^{-1}\}\bar{A}(\mathbf{p'}\omega') \right]. \quad (5.90)$$

The latter approaches zero when the denominator, $\omega+\omega'-\bar{E}_n$ does not vanish for any choice of ω' and $\bar{E}_n$ for which the numerator is nonvanishing. On the other hand, if $\bar{E}_n-\omega'=\omega$ for some ω' and $\bar{E}_n$ for which the numerator is finite, the limit is

$$\Gamma(\mathbf{p}\omega) = \int \frac{d\mathbf{p'}}{(2\pi)^3} \int \frac{d\omega'}{2\pi} 2\pi \sum_n' \delta(\bar{E}_n-\omega-\omega')\bar{r}_n(\mathbf{pp'})$$

$$\times \left(\frac{1}{e^{i\omega\tau}-1} \mp \frac{1}{e^{i(\omega+\omega')\tau}-1} \right)\bar{A}(\mathbf{p'}\omega'). \quad (5.91)$$

The prime on the summation indicates that it contains only those poles, $\bar{E}_n$, which lie on the real axis, since these are the only ones for which limit of the second term of (5.90) is nonvanishing. Special consideration must be given to any real ω_0 which satisfies $\omega_0 = \epsilon(\mathbf{p},\omega_0)$ and $\Gamma'(\omega_0) = 2\delta \to 0$ since at such a point, (5.86) becomes $\bar{A}(\mathbf{p}\omega) = 2\pi\delta(\omega - \epsilon(\mathbf{p}\omega))$. This special case applies, for example, to the noninteracting system.

Equations (5.86) to (5.89) may also be derived by using the summation techniques employed with (5.31). When $t>t'$ we infer from (5.82) that

$$\bar{G}_1(\mathbf{p}, t-t') = \int_c \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{(1 \mp e^{-i\omega\tau})[\omega-h-\bar{V}(\mathbf{p},\omega)+\zeta]},$$

(5.92)

where the contour of integration is chosen to pass above the singularities of $[\omega-h+\zeta-\bar{V}(\mathbf{p},\omega)]^{-1}$ (which lie on the real axis since they represent the possible energy differences of the system) and below the poles of $(1 \mp e^{-i\omega\tau})$. By closing the contour in the upper half plane so that it encircles in a counterclockwise sense the points at which $(1 \mp e^{-i\omega\tau})$ vanishes, we obtain the summation (5.77), provided $0<t-t'<\tau$. By encircling the singularities of $[\omega-h+\zeta-\bar{V}(\mathbf{p},\omega)]^{-1}$ we obtain $\bar{G}$ for all $t-t'>0$. Comparison of the latter expression with (3.26) leads to the relations (5.86), (5.87), and (5.89).

We note that expressions of the form (5.86) and (5.92) follow for the rigorous single-particle function as well as the approximate Green's function discussed above. Consequently these equations and those subsequent ones which do not employ the forms (5.83), (5.88) and (5.90) for V are generally valid.

If we replace $i\tau$ by β and ζ by μ, we obtain for the real function $A^{\alpha\beta}(\mathbf{p},\omega)$

$$A^{\alpha\beta}(\mathbf{p}\omega) = \frac{\Gamma^{\alpha\beta}(\mathbf{p}\omega)}{[\omega-\epsilon^{\alpha\beta}(\mathbf{p}\omega)]^2 + [\frac{1}{2}\Gamma^{\alpha\beta}(\mathbf{p}\omega)]^2},$$

(5.93)

where

$$\epsilon^{\alpha\beta}(\mathbf{p},\omega) = \frac{\mathbf{p}^2}{2m} + \mathrm{Re}\, V^{\alpha\beta}(\mathbf{p},\omega\pm i0),$$

(5.94)

and

$$\Gamma^{\alpha\beta}(\mathbf{p}\omega) = 2\,\mathrm{Im}\, V^{\alpha\beta}(\mathbf{p}, \omega-i0).$$

(5.95)

When (5.88) and (5.90) are introduced, the functions $\epsilon^{\alpha\beta}(\mathbf{p}\omega)$ and $\Gamma^{\alpha\beta}(\mathbf{p}\omega)$ become

$$\epsilon^{\alpha\beta}(\mathbf{p}\omega) = \frac{\mathbf{p}^2}{2m} + \int \frac{d\mathbf{p}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi}$$
$$\times \mathrm{Re}[\langle \tfrac{1}{2}(\mathbf{p}-\mathbf{p}') | t^{\alpha\beta}(\mathbf{p}+\mathbf{p}', \omega+\omega' \pm i0)$$
$$\pm t'^{\alpha\beta}(\mathbf{p}+\mathbf{p}', \omega+\omega' \pm i0) | \tfrac{1}{2}(\mathbf{p}-\mathbf{p}')\rangle]$$
$$\times \frac{A^{\alpha\beta}(\mathbf{p}'\omega')}{e^{\alpha+\beta\omega'}\mp 1} \mp \int \frac{d\mathbf{p}'}{(2\pi)^3} \mathrm{P} \int \frac{d\omega'}{2\pi}$$
$$\times \sum_n \frac{r_n^{\alpha\beta}(\mathbf{p}\mathbf{p}')}{\omega+\omega'-E_n^{\alpha\beta}} \frac{A^{\alpha\beta}(\mathbf{p}'\omega')}{e^{2\alpha+\beta E_n}-1}$$

(5.96)

and

$$\Gamma^{\alpha\beta}(\mathbf{p}\omega) = \int \frac{d\mathbf{p}'}{(2\pi)^3} \int \frac{d\omega'}{2\pi}$$
$$\times 2\,\mathrm{Im}\langle \tfrac{1}{2}(\mathbf{p}-\mathbf{p}') | t^{\alpha\beta} \pm t'^{\alpha\beta} | \tfrac{1}{2}(\mathbf{p}-\mathbf{p}')\rangle$$
$$\times \left[\frac{1}{e^{\alpha+\beta\omega'}\mp 1} \mp \frac{1}{e^{2\alpha+\beta(\omega+\omega')}-1}\right] A^{\alpha\beta}(\mathbf{p}'\omega').$$

(5.97)

In these expressions, we have introduced the function $t^{\alpha\beta}(\mathbf{P},\omega) = t^{\alpha\beta}(\mathbf{P}, \nu = (\omega-2\mu)\tau/\pi)$ and utilized the relation $t^{\alpha\beta}(z^*) = t^{\alpha\beta}(z)^*$ which follows from the reality of (5.69) when z is real.

As the notation is intended to suggest, the quantities ϵ and Γ have simple interpretations, at least in the neighborhood of those points $\omega_0(\mathbf{p})$ for which

$$\epsilon(\mathbf{p},\omega_0) = \omega_0 \quad \text{and} \quad (d\Gamma/d\omega)_{\omega_0} \ll [1-(d\epsilon/d\omega)_{\omega_0}]. \quad (5.98)$$

In the neighborhood of these points we may write

$$A^{\alpha\beta}(\mathbf{p}\omega) = \frac{Z(\omega_0)\gamma(\omega_0)}{(\omega-\omega_0)^2 + [\frac{1}{2}\gamma(\omega_0)]^2}, \quad (5.99)$$

where

$$Z = \gamma/\Gamma = [1-(d\epsilon/d\omega)_{\omega_0}]^{-1}. \quad (5.100)$$

When (5.99) continues to be a good approximation to $A(\mathbf{p}\omega)$ for values of $\omega-\omega_0$ which are sufficiently large multiples of $\gamma(\omega_0)$, the frequency integral of $A(\mathbf{p}\omega)$ in this neighborhood equals $2\pi Z(\omega_0)$. Since the entire frequency integral of $A(\mathbf{p}\omega)$ has the value 2π, a necessary condition for the dominance of the region is that $Z(\omega_0) \approx 1$, or equivalently that $|d\epsilon/d\omega| \ll 1$. For fermions, the additional restriction $A(\mathbf{p}\omega) \geqslant 0$ insures that there can be only one such point and that at this point $Z(\omega_0) < 1$ and $d\epsilon/d\omega < 0$. The limiting case of this behavior arises when $A(\mathbf{p}\omega)$ is rigorously $2\pi\delta(\omega-\omega_0(\mathbf{p}))$. We shall call any solution of $\omega_0 = \epsilon(\mathbf{p},\omega_0)$ a resonance and $\gamma(\omega_0)$ the resonance width. When the statistical factor in the expression for $G_1^{\alpha\beta}(\mathbf{p},t)$ also varies slowly in the neighborhood of its resonances, their contribution to G_1 is given by

$$\pm iG_1^{\alpha\beta}(\mathbf{p}t) = \sum_i Z_i\{\pm 1 + [\exp(\alpha+\beta\omega_i(\mathbf{p})) \mp 1]^{-1}\}$$
$$\times e^{-i\omega_i(\mathbf{p})t - \frac{1}{2}\gamma_i t}, \quad t>0,$$
$$= \sum_i Z_i[\exp(\alpha+\beta\omega_i(\mathbf{p})) \mp 1]^{-1}$$
$$\times e^{-i\omega_i(\mathbf{p})t - \frac{1}{2}\gamma_i|t|}, \quad t<0.$$

(5.101)

We have here allowed for the possibility of several resonances, whose weights in the fermion example must satisfy $\sum_i Z_i < 1$. Equation (5.101) exhibits the relation between the widths of single-particle resonances and the corresponding lifetimes. These lifetimes reflect the approximate rate at which equilibrium is restored when a particle of momentum $\mathbf{p}$ is suddenly introduced or removed from the medium. We shall not enter here into a detailed discussion of the dominance of (5.101) but merely observe that it can be the major contribution only for a restricted time interval. This interval cannot extend to very short times since the discontinuity at $t=0$ of the approximate form (5.101) does not generally reproduce the unit discontinuity of iG. This error in the discontinuity reflects the importance of the entire frequency spectrum of $A(\mathbf{p},\omega)$ for very short times. The influence of the complete spectrum is only transient, however, the nonresonant contribution decreasing rapidly with time. Nevertheless, at sufficiently long times, this contribution overshadows the

exponentially decreasing term (5.101). As long as $\gamma(\omega_0)$ is not exactly zero, the resonances will have lifetimes characteristic of the interaction strength and the particle density. If $\gamma(\omega_0) \to 0$ when $\delta \to 0$, that is, if the second term in (5.90) vanishes identically, the lifetimes will be limited only macroscopically and superfluid properties will result.

In the low-density limit $(\alpha \to \infty)$, Γ reduces to the anticipated collision rate. The function Γ is then expressed in terms of the imaginary part of the diagonal element of the scattering matrix t. The latter is related to the total cross section σ by the optical theorem which states that $\sigma |\mathbf{p}-\mathbf{p}'|/2m = \mathrm{Im}\langle\frac{1}{2}(\mathbf{p}-\mathbf{p}')|t \pm t'|\frac{1}{2}(\mathbf{p}-\mathbf{p}')\rangle$. In terms of the velocity, $\mathbf{v} = \mathbf{p}/m$, of the particles we obtain the elementary low-density result

$$\Gamma(\mathbf{p}) = \int \frac{d\mathbf{p}'}{(2\pi)^3}\sigma(|\mathbf{v}-\mathbf{v}'|)|\mathbf{v}-\mathbf{v}'|n(\mathbf{p}'). \quad (5.102)$$

The expressions (5.96) and (5.97) also contain a term which indicates through its characteristic statistical denominator $[\exp(2\alpha+\beta(\omega+\omega'))-1]^{-1}$ the Bose nature of a composite system formed by two fermions or bosons. This term also contributes to the width when we depart from the limit in which $\alpha \to \infty$.

We note, finally, that Eqs. (5.94) and (5.48) may be used to test the validity of a single-level approximation

$$A(\mathbf{p},\omega) = 2\pi\delta(\omega-\omega_0(\mathbf{p})). \quad (5.103)$$

With this weight function, Eq. (5.94) implies that

$$\omega_0(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{p},\omega_0(\mathbf{p}))$$

$$= \frac{\mathbf{p}^2}{2m} + \int \frac{d\mathbf{p}'}{(2\pi)^3}\langle|v \pm v'|\rangle n(\mathbf{p}')$$

$$+ \mathrm{P}\int \frac{d\mathbf{p}'}{(2\pi)^3}\sum_n\left[\frac{1}{e^{\alpha+\beta\omega_0(\mathbf{p}')} \mp 1} \mp \frac{1}{e^{2\alpha+\beta E_n}-1}\right]$$

$$\times \frac{r_n^{\alpha\beta}(\mathbf{p},\mathbf{p}')}{\omega_0(\mathbf{p})+\omega_0(\mathbf{p}')-E_n}. \quad (5.104)$$

Similarly the relation (5.48) for the mean value of the frequency yields

$$\omega_0(\mathbf{p}) \to \frac{\mathbf{p}^2}{2m} + \int \frac{d\mathbf{p}'}{(2\pi)^3}\langle|v \pm v'|\rangle n(\mathbf{p}'). \quad (5.105)$$

The difference of these expressions is therefore a measure of the dispersion in A. What frequency, if any, can be approximately associated with a given momentum excitation when (5.104) and (5.105) differ, depends upon the time during which that excitation is important. This is reflected in the investigations which

led to (5.94) and (5.48). In the latter case we considered high frequencies and correspondingly short times; in the former we were concerned with frequency poles and long time behavior.

Considerations similar to these apply to the multiparticle Green's functions, and in particular, to the special density correlation function which we described in Sec. IV. Thus, when σ is a continuous function the conductivity is determined microscopically, while when σ contains a δ function the system exhibits superconductivity.

VI. FORMAL SOLUTIONS. OTHER APPROXIMATIONS

In previous sections we characterized the Green's functions of the system by an infinite sequence of coupled equations and showed that numerous properties of the system could be obtained from approximate solutions of the first few of these. We turn now to the more general investigation of these equations and their solutions. We shall carry out this analysis with more powerful field-theoretical techniques, which permit concise expression and formal solution of the Green's function equations. From these solutions we shall then derive alternative methods of systematic approximation.

In order to express the Green's function equations in compact form, it is convenient to introduce arbitrary functions of space and time $\xi(\mathbf{r}t)$ and $\eta(\mathbf{r}t)$, called source functions, which are completely commutative when the ψ field obeys Bose statistics and completely anticommutative when the ψ field obeys Fermi statistics. By means of these functions, a Green's functional may be defined:

$$\bar{G}[\xi,\eta] = 1 + \sum_1^\infty \int \frac{\xi(n)\cdots\xi(1)}{n!}$$

$$\times \bar{G}_n(1,\cdots n;1'\cdots n')\frac{\eta(1')\cdots\eta(n')}{n!}. \quad (6.1)$$

In this equation, we have employed the conventions of Sec. V. Thus, the integration sign applies to all $2n$ coordinates and extends over all space and the time interval $[0,\tau]$. The individual Green's functions are thus characterized as coefficients in the expansion of the functional, or equivalently as its functional derivatives:

$$\frac{\delta_l}{\delta\eta(n')}\cdots\frac{\delta_l}{\delta\eta(1')}\frac{\delta_l}{\delta\xi(1)}\cdots\frac{\delta_l}{\delta\xi(n)}\bar{G}[\xi,\eta]\Big|_{\eta=\xi=0}$$

$$= \bar{G}_n(1\cdots n;1'\cdots n'). \quad (6.2)$$

The distinction between left and right variational derivatives which is necessary for the anticommutative fermion source functions is indicated by the subscript l or r. With the aid of (6.2), it is easy to verify that the set of Green's function equations (5.4) is equivalent to

the single functional differential equation

$$\left[\left(i\frac{\partial}{\partial t_1}-h_1+\zeta\right)\frac{\delta_l}{\delta\xi(1)}\mp i\int v(1,2)\frac{\delta_l}{\delta\eta(2^+)}\frac{\delta_l}{\delta\xi(2)}\right.$$

$$\left.\times\frac{\delta_l}{\delta\xi(1)}-\eta(1)\right]\bar{G}[\xi,\eta]=0,\quad(6.3)$$

or its adjoint,

$$\left[\left(-i\frac{\partial}{\partial t_1}-h_1+\zeta\right)\frac{\delta_r}{\delta\eta(1)}\mp i\int v(1,2)\frac{\delta_r}{\delta\xi(2^-)}\frac{\delta_r}{\delta\eta(2)}\right.$$

$$\left.\times\frac{\delta_r}{\delta\eta(1)}-\xi(1)\right]\bar{G}[\xi,\eta]=0.\quad(6.4)$$

In the interval $[0,\tau]$ to which (6.3) and (6.4) are restricted, the boundary condition (5.17) specifies the particular solution of the differential equation in time. The additional condition necessary for the normalization of these equations is determined from (6.1) by the relation

$$\bar{G}[0,0]=1.\quad(6.5)$$

A simple illustration of the functional technique is afforded by the direct integration of the differential equation in the trivial example with no interaction. In that case we obtain

$$\bar{G}[\xi,\eta]=\exp(\xi\bar{G}_1{}^0\eta)\quad(6.6)$$

for the Green's functional. This functional serves to generate the multiparticle Green's functions in the form

$$\bar{G}_n{}^0=\text{perm}_{(n)}\text{ or }\det_{(n)}\bar{G}_1{}^0,\quad(6.7)$$

where $\text{perm}_{(n)}$ and $\det_{(n)}$ are the permanent and determinant of the $n\times n$ coordinate indexed matrix $\bar{G}_1{}^0(i;j')$.

A successive correlation approximation for treating interactions may be introduced within this framework. As a first approximation we take a Green's functional which contains no dynamical correlations,

$$\bar{G}[\xi,\eta]=\exp[\xi\bar{G}_1\eta].\quad(6.8)$$

We then determine the function $\bar{G}_1$ so that Eq. (6.3) is satisfied to the lowest order in ξ and η. The equation for $\bar{G}_1$ thus obtained is the expected one,

$$[i(\partial/\partial t_1)-h_1+\zeta]\bar{G}_1(1;1')$$

$$\mp i\int v(1,2)[\bar{G}_1(1;1')\bar{G}_1(2;2^+)$$

$$\pm\bar{G}_1(1;2^+)\bar{G}_1(2;1')]=\delta(1,1').\quad(6.9)$$

As a second approximation we take an exponential form which reproduces correctly the first two powers of ξ and η,

$$\bar{G}[\xi,\eta]=\exp(\xi\bar{G}_1\eta)$$
$$\times\exp[(2!)^{-1}\xi\xi(\bar{G}_2-2\bar{G}_1\bar{G}_1)\eta\eta(2!)^{-1}],\quad(6.10)$$

and determine the pair of functions by requiring that the coefficients of the two lowest powers of ξ and η in the functional equation (6.3) vanish. As a third approximation we would include three-particle correlations, determining the coefficient functions in a functional

$$\bar{G}[\xi,\eta]=\exp(\xi\bar{G}_1\eta)\exp[(2!)^{-1}\xi\xi(\bar{G}_2-2\bar{G}_1\bar{G}_1)\eta\eta(2!)^{-1}]$$
$$\times\exp[(3!)^{-1}\xi\xi\xi(\bar{G}_3-9\bar{G}_2\bar{G}_1+12\bar{G}_1\bar{G}_1\bar{G}_1)$$
$$\times\eta\eta\eta(3!)^{-1}],\quad(6.11)$$

by satisfying (6.3) to third orders in ξ and η; and so on.

The pair of equations for $\bar{G}_1$ and $\bar{G}_2$ which result from (6.10) are readily determined. The first equation is identical with (5.3),

$$[i(\partial/\partial t_1)-h_1+\zeta]\bar{G}_1(1;1')$$

$$\mp i\int v(1,2)\bar{G}_2(12;1'2^+)=\delta(1,1'),\quad(6.12)$$

and would be exact if the correct $\bar{G}_2$ were inserted. The second equation is

$$[i(\partial/\partial t_1)-h_1+\zeta]\bar{G}_2(12;1'2')$$

$$\mp i\int v(1,3)[\bar{G}_2(13;1'3^+)\bar{G}_1(2;2')$$

$$+\bar{G}_2(13;3^+2')\bar{G}_1(2;1')+\bar{G}_2(13;2'1')\bar{G}_1(2;3^+)]$$

$$\mp i\int v(1,3)\{\bar{G}_2(32;1'3^+)-[\bar{G}_1(3;1')\bar{G}_1(2;3^+)$$

$$\pm\bar{G}_1(3;3^+)\bar{G}_1(2;1')]\}\bar{G}_1(1;2')$$

$$+\text{cyclic permutations of }1',2',3^+$$

$$\mp i\int v(1,3)\{\bar{G}_2(21;1'3^+)-[\bar{G}_1(2;1')\bar{G}_1(1;3^+)$$

$$\pm\bar{G}_1(2;3^+)\bar{G}_1(1;1')]\}\bar{G}_1(3;2')$$

$$+\text{cyclic permutations of }1'2'3^+$$

$$=\delta(1,1')\bar{G}_1(2;2')\pm\delta(1,2')\bar{G}_1(2;1').\quad(6.13)$$

For weak interactions or low densities the last two sets of terms on the left-hand side of (6.13), which contain the interaction of one pair and the correlation of another, are relatively unimportant. If these terms are omitted, the equation may be re-expressed as

$$[i(\partial/\partial t_1)-h_1+\zeta]\{\bar{G}_2(12;1'2')$$

$$-[\bar{G}_1(1;1')\bar{G}_1(2;2')\pm\bar{G}_1(1;2')\bar{G}_1(2;1')]\}$$

$$-i\int\bar{G}_1(2;2'')v(1,2'')\bar{G}_2(12'';1'2')=0.\quad(6.14)$$

which we encountered as (5.55). Application of a low-

density or weak-interaction approximation again leads to the replacement of $\bar{G}_1(2; 2'')$ by $\bar{G}_1{}^0(2; 2'')$ and thus to (5.57). We note that terms suggestive of an improved equation, in which the converse replacement is partially introduced, are included in (6.13). Thus, one term in (6.13) is

$$\mp \int v(1,3)G(3; 3^+)\{\bar{G}_2(12; 1'2')$$

$$-[\bar{G}(1; 1')\bar{G}_1(2; 2')\pm\bar{G}_1(1; 2')\bar{G}_1(2; 1')]\}, \quad (6.15)$$

which is the direct interaction term that must be added to $(\bar{G}_1{}^0)^{-1}$ in a Hartree approximation for $G_1{}^{-1}$. With no further approximations, Eqs. (6.12) and (6.13) represent a second step in a systematic method whose first step is the Hartree approximation. In general, each step of this procedure differs from the corresponding one in the previously outlined scheme through the method of approximating the $(n+1)$-particle Green's function in the first n coupled equations. In (6.13), unlike (5.54), the arguments of the Green's function in the interaction term are treated symmetrically. Indeed, it is the terms in (6.13) not present in (5.54) which permit discussion of long range collective motions.

The great advantage of the functional formulation (6.3) lies in the possibility of obtaining formal expressions for the multiparticle Green's functions and for the trace. One form for the formal solution of the differential equation is easily obtained. We note that the equation contains η and a differential operator which can be expressed as $[\Delta,\eta]$, where Δ is defined by

$$\Delta \equiv \left\{ \mp \int \frac{\delta}{\delta\eta(1^+)}\left(i\frac{\partial}{\partial t}+\zeta-h\right)\frac{\delta}{\delta\xi(1)} \right.$$

$$\left. +\frac{i}{2}\int \frac{\delta}{\delta\eta(1^+)}\frac{\delta}{\delta\eta(2^+)}v(1,2)\frac{\delta}{\delta\xi(2)}\frac{\delta}{\delta\xi(1)} \right\}. \quad (6.16)$$

Since $[\Delta,[\Delta,\eta]]=0$, we may write Eq. (6.3) in the form

$$e^\Delta\eta e^{-\Delta}\bar{G}[\xi,\eta]=0, \quad (6.17)$$

and infer that $\eta e^{-\Delta}\bar{G}$ vanishes. From this equation and the analogous one for ξ we obtain

$$\bar{G}[\xi,\eta]=e^\Delta\delta[\xi,\eta], \quad (6.18)$$

where $\delta[\xi,\eta]$ is a functional which satisfies $\eta\delta[\xi,\eta]=0$ and $\xi\delta[\xi,\eta]=0$.

We arrive at a more useful expression for the Green's functional by a rearrangement of (6.18) which utilizes the possibility of describing the effect of the interaction potential v in terms of the behavior of the system in the presence of an external, single-particle, generally spatially nonlocal potential $U(11')$. Using a four-dimensional matrix notation in which the time dependence is given by $\delta(t_1-t_1'-0^+)$ and defining $\bar{G}[\xi,\eta,U]$ in the

presence of this potential by

$$\bar{G}[\xi,\eta,U]=e^{\Delta[U]}\delta[\xi,\eta],$$

$$\Delta[U]=\Delta\pm\int \frac{\delta}{\delta\eta(1)}U(11')\frac{\delta}{\delta\xi(1')}, \quad (6.19)$$

we obtain the identity

$$\pm\frac{\delta}{\delta\eta(1)}\frac{\delta}{\delta\xi(1')}\bar{G}[\xi,\eta,U]=\frac{\delta}{\delta U(11')}\bar{G}[\xi,\eta,U]. \quad (6.20)$$

The description of the interaction effects is transferred from the source dependence to the potential dependence by using this relation to replace

$$\int \frac{\delta}{\delta\eta(1^+)}\frac{\delta}{\delta\eta(2^+)}v(1,2)\frac{\delta}{\delta\xi(2)}\frac{\delta}{\delta\xi(1)}$$

$$\text{by} \quad \int \frac{\delta}{\delta U(1)}v(1,2)\frac{\delta}{\delta U(2)} \quad (6.21)$$

in (6.19). (A local potential U has been employed to simulate the local interaction v.) We then have the expression

$$\bar{G}[\xi,\eta,U]=\exp\left(\frac{i}{2}\frac{\delta}{\delta U}v\frac{\delta}{\delta U}\right)$$

$$\times\exp\left[\mp\frac{\delta}{\delta\eta}\left(i\frac{\partial}{\partial t}-h+\zeta-U\right)\frac{\delta}{\delta\xi}\right]\delta[\xi,\eta]. \quad (6.22)$$

The effect of the source derivatives on the δ-functional is readily obtained since it is, apart from a functional of U, the Green's functional for a noninteracting system subject to an external potential. This functional is the analog of (6.6) with $\bar{G}_1{}^0$ replaced by $\bar{G}_1{}^0[U]$,

$$\bar{G}_1{}^0[U]=[i(\partial/\partial t)-h+\zeta-U]^{-1}$$
$$=[1-\bar{G}_1{}^0U]^{-1}\bar{G}_1{}^0, \quad (6.23)$$

and therefore it follows that

$$\exp\left[\mp\frac{\delta}{\delta\eta}\left(i\frac{\partial}{\partial t}-h+\zeta-U\right)\frac{\delta}{\delta\xi}\right]\delta[\xi,\eta]$$

$$=D[U]\exp(\xi\bar{G}_1{}^0[U]\eta), \quad (6.24)$$

where $D[U]$ is a functional of U. A differential equation for this functional is obtained by applying the relation (6.20) to (6.24) and setting $\xi=\eta=0$,

$$\delta D[U]=\pm\int D[U]\bar{G}_1{}^0[11'; U]\delta U(1'1). \quad (6.25)$$

By means of (6.23) this equation may be written in the form

$$\delta \ln D[U]=\mp\text{tr}[(1-\bar{G}_1{}^0U)^{-1}\delta(1-\bar{G}_1{}^0U)], \quad (6.26)$$

where tr indicates diagonal summation of the coordinate-indexed matrix. In view of the relation

$$\delta \ln \det X = \operatorname{tr}(X^{-1}\delta X), \quad (6.27)$$

the solution of (6.25) is, apart from a constant,

$$D[U] = \det^{\mp 1}(1 - \bar{G}_1{}^0 U). \quad (6.28)$$

An equivalent expression is obtained by letting $U \to \kappa U$ and $\delta U \to d\kappa U$ in (6.26)

$$d \ln D[U] = \pm \operatorname{tr}[(1 - \kappa \bar{G}_1{}^0 U)^{-1}\bar{G}_1{}^0 U d\kappa], \quad (6.29)$$

whence, on integrating from $\kappa = 0$ to $\kappa = 1$, we obtain

$$D[U] = \exp[\mp \operatorname{tr} \ln(1 - \bar{G}_1{}^0 U)]. \quad (6.30)$$

Apart from a constant, the functional $\bar{G}[\xi, \eta, U]$ is therefore given by

$$\bar{G}[\xi, \eta, U] = \exp\left(\frac{i}{2}\frac{\delta}{\delta U}v\frac{\delta}{\delta U}\right) \det^{\mp 1}(1 - \bar{G}_1{}^0 U)$$
$$\times \exp(\xi \bar{G}_1{}^0[U]\eta). \quad (6.31)$$

In particular, to within a constant,

$$\bar{G}[0, 0, U] \equiv \exp(\mathcal{W}[U])$$
$$= \exp\left(\frac{i}{2}\frac{\delta}{\partial U}v\frac{\delta}{\delta U}\right) \det^{\mp 1}(1 - \bar{G}_1{}^0 U). \quad (6.32)$$

If we redefine the constant in $\bar{G}[\xi, \eta, U]$ so that

$$\bar{G}[\xi, \eta, U] = \exp(-\mathcal{W}[U]) \exp\left(\frac{i}{2}\frac{\delta}{\partial U}v\frac{\delta}{\delta U}\right)$$
$$\times \det^{\mp 1}(1 - \bar{G}_1{}^0 U) \exp(\xi \bar{G}_1{}^0[U]\eta), \quad (6.33)$$

we see that this expression satisfies the analog of (6.5), and that its expansion in sources generates the Green's functions

$$\bar{G}_n[1, \cdots n; 1' \cdots n'; U]$$

$$= \exp(-\mathcal{W}[U]) \exp\left(\frac{i}{2}\frac{\delta}{\delta U}v\frac{\delta}{\delta U}\right) \det^{\mp 1}(1 - \bar{G}_1{}^0 U)$$
$$\times \operatorname{perm}_{(n)} \text{ or } \det_{(n)} \bar{G}_1{}^0[i, j'; U]. \quad (6.34)$$

When U is set equal to zero, these functions become the previously defined Green's functions.

We turn now to the determination of the trace, $\exp W$. The form of the Green's functions suggest that $\exp \mathcal{W}[U]$ is related to the trace, $\exp W$. This relation is made precise by introducing a functional $W[U]$ which satisfies the analog of (3.7),

$$\frac{\partial}{\partial \zeta}W[U] = \mp \operatorname{tr}\bar{G}_1[U]. \quad (6.35)$$

In this expression the trace is the sum of diagonal matrix elements obtained by letting the left time index approach the right time index from earlier times. The corresponding derivative of $\mathcal{W}$ is given by

$$\frac{\partial \mathcal{W}}{\partial \zeta} = \mp \exp(-\mathcal{W}[U]) \exp\left(\frac{i}{2}\frac{\delta}{\delta U}v\frac{\delta}{\delta U}\right)$$
$$\times \det^{\mp 1}(1 - \bar{G}_1{}^0 U) \operatorname{tr}[(1 - \bar{G}_1{}^0 U)^{-1}\bar{G}_1{}^0\bar{G}_1{}^0 U], \quad (6.36)$$

which, by (6.23), is equivalent to

$$\mp \exp(-\mathcal{W}[U]) \exp\left(\frac{i}{2}\frac{\delta}{\delta U}v\frac{\delta}{\delta U}\right)$$
$$\times \det^{\mp 1}(1 - \bar{G}_1{}^0 U) \operatorname{tr}(\bar{G}_1{}^0[U] - \bar{G}_1{}^0). \quad (6.37)$$

Substitution of (6.34) in (6.37) leads to an expression analogous to (6.35), and thus to the relation

$$(\partial/\partial \zeta)(W - \mathcal{W}) = \mp \operatorname{tr} \bar{G}_1{}^0. \quad (6.38)$$

The right-hand side of Eq. (6.38) may be replaced by its known value (5.27),

$$\mp \operatorname{tr}\bar{G}_1{}^0 = i\tau \operatorname{tr}^{(3)}[\exp(i(h - \zeta)\tau)\mp 1]^{-1}$$
$$= \mp \frac{d}{d\zeta} \operatorname{tr}^{(3)} \ln[1 \mp \exp(-i(h - \zeta)\tau)], \quad (6.39)$$

where $\operatorname{tr}^{(3)}$ is the trace of the spatial coordinate matrix. Integrating with respect to ζ and determining the constant by noting that W, $\mathcal{W}$, and $\operatorname{tr}^{(3)}$ all approach zero when $\zeta \to i\infty$ (effectively the limit of zero particle number), we finally obtain

$$\exp W = \exp W_0 \exp\left(\frac{i}{2}\frac{\delta}{\delta U}v\frac{\delta}{\delta U}\right) \det^{\mp 1}(1 - \bar{G}_1{}^0 U), \quad (6.40)$$

or its equivalent

$$\exp W = \exp W_0 \exp\left(\frac{i}{2}\frac{\delta}{\delta U}v\frac{\delta}{\delta U}\right)$$
$$\times \exp[1 \mp \operatorname{tr} \ln(1 - \bar{G}_1{}^0 U)]. \quad (6.41)$$

These expressions identify $\exp W_0$,

$$\exp W_0 = \det^{(3)\mp 1}(1 \mp e^{-i(h - \zeta)\tau})$$
$$= \prod_\alpha [1 \mp \exp(-i(h_\alpha - \zeta)\tau)]^{\mp 1}, \quad (6.42)$$

as the trace for a system with no interaction.

Just as the occurrence of $\exp[\pm i(\delta/\delta\eta)U(\delta/\delta\xi)]$ in (6.19) allowed conversion of source derivatives to external potential derivatives (6.20), the appearance of $\exp[(i/2)(\delta/\delta U)v(\delta/\delta U)]$ permits the replacement of external potential derivatives by interaction potential derivatives. By employing the relation

$$\frac{\delta}{\delta v(12)}e^W = \frac{i}{2}\frac{\delta}{\delta U(2)}\frac{\delta}{\delta U(1)}e^W, \quad (6.43)$$

and the identity that follows from (6.2) and (6.20),

$$[\delta/\delta U(n+1)]e^W \bar{G}_n[1,\cdots n;1'\cdots n',U]$$
$$=\pm e^W \bar{G}_{n+1}[1,\cdots n+1;1'\cdots(n+1)^+;U], \quad (6.44)$$

we obtain

$$\delta W/\delta v(12)=\tfrac{1}{2}i\bar{G}_2(12;1^+2^+). \quad (6.45)$$

Equations (6.43) and (6.44) may also be employed to evaluate higher derivatives of W. For its second derivative, they yield

$$\frac{\delta^2 W}{\delta v(1,2)\delta v(3,4)}=e^{-W}\frac{\delta^2}{\delta v(1,2)\delta v(3,4)}e^W-\frac{\delta W}{\delta v(1,2)}\frac{\delta W}{\delta v(3,4)}$$

$$=(\tfrac{1}{2}i)^2[\bar{G}_4(1234;1^+2^+3^+4^+)$$
$$-\bar{G}_2(12;1^+2^+)\bar{G}_2(34;3^+4^+)], \quad (6.46)$$

and for the higher derivatives, they generate analogous but successively longer expressions.

If we consider variations in the strength of an interaction potential κv, we may specialize (6.45) to

$$\frac{dW(\kappa)}{d\kappa}=\tfrac{1}{2}i\int v(1,2)\bar{G}_2^\kappa(12;1^+2^+), \quad (6.47)$$

where $\exp W(\kappa)$ is the trace and $\bar{G}^\kappa$ the Green's functions for the system with potential κv. Since $W(\kappa)$ is proportional to the occupied volume for all potential strengths κv, the right-hand side depends linearly on the volume. The expressions for the higher derivatives of W are likewise linearly dependent on the volume. On the other hand, individual terms in the expression for $d^n W/d\kappa^n$ need not be proportional to the volume. In particular, the various terms subtracted from the first one encountered in the evaluation of $d^n W/d\kappa^n$ by the procedure (6.46),

$$e^{-W(\kappa)}\frac{d^n}{d\kappa^n}e^{W(\kappa)}=e^{-W}\int v\cdots v\frac{\delta^n}{\delta v^n}e^W$$

$$=(\tfrac{1}{2}i)^n\int v(1,2)\cdots v(2n-1,2n)$$

$$\times\bar{G}_{2n}^\kappa(1,\cdots 2n;1^+,\cdots 2n^+), \quad (6.48)$$

serve to eliminate its dependence on the second to nth powers of the volume.

In the special case $\kappa=0$, all derivatives may be expressed in terms of products of noninteracting single-particle Green's functions. Under this condition, we verify by induction that computation of the nth derivative of W involves the subtraction from

$$\bar{G}_{2n}^0=\text{perm}_{(2n)} \text{ or } \det_{(2n)}\bar{G}^0(i;j),$$

of those terms and only those terms

$$\bar{G}_1^0(1;i_1)\bar{G}_1^0(2;i_2)\cdots\bar{G}_1^0(2n;i_{2n}) \quad (6.49)$$

whose arguments i_k admit a partition in which i_1, i_2, $\cdots i_{2l}\leqslant 2l$ for some $l<n$. The eliminated terms are the ones whose integrals

$$\int v(1,2)\cdots v(2n-1,2n)\bar{G}_1^0(1;i_1)\cdots$$
$$\times\bar{G}_1^0(2n;i_{2n}) \quad (6.50)$$

may be factored and therefore behave as powers of the volume higher than the first; the remaining ones yield integrals depending linearly on the volume. We designate the latter terms as the connected part $C_{2n}^0(1,\cdots 2n;1,\cdots 2n)$ of $\bar{G}_{2n}^0(1,\cdots 2n;1^+,\cdots 2n^+)$. If the interaction potential is one which admits a Taylor series expansion of $W-W_0$ about zero potential, the combinations C_{2n}^0 which characterize the potential strength derivatives of $W-W_0$ provide the expansion,

$$W-W_0=\sum_1^\infty \frac{1}{n!}\int\left(\frac{i}{2}\right)^n v(1,2)\cdots$$
$$\times v(2n-1,2n)C_{2n}^0(1,\cdots 2n;1,\cdots 2n). \quad (6.51)$$

By setting $\tau=-i\beta$, $\zeta=-\alpha/\beta$, we obtain for the pressure, particle density, and energy per unit volume, expansions in powers of the potential which are explicitly independent of the volume. Special examples of expansions of this character (linked cluster expansions) have been derived from perturbation theory by several authors. Unfortunately, such expansions of W, which imply similar expansions of all the Green's functions, are not generally permissible.

We continue the more general discussion by recasting the expressions for the Green's functions in differential form. For this purpose we employ the relation

$$\exp\left(\frac{i}{2}\frac{\delta}{\delta U}v\frac{\delta}{\delta U}\right)f(U)\exp\left(-\frac{1}{2}\frac{\delta}{\delta U}v\frac{\delta}{\delta U}\right)$$
$$=f\left(U+iv\frac{\delta}{\delta U}\right) \quad (6.52)$$

and Eq. (6.32) to write (6.34) in the form

$$\bar{G}_n=e^{-W[U]}\frac{\text{perm}_{(n)}}{\det_{(n)}}\bar{G}_1^0\left[i,j';U+iv\frac{\delta}{\delta U}\right]e^{W[U]}. \quad (6.53)$$

From this expression we derive differential equations for Green's functions by applying

$$(i\partial/\partial t+\zeta-h-U-iv\delta/\delta U)$$

to $e^W\bar{G}_n$ and using the identity (6.44),

$$[(\bar{G}_1^0)^{-1}-U]_1\bar{G}_n\mp i\int v(1,n+1)\bar{G}_{n+1}$$
$$=\sum_{i'}(-1)^{i'-1}\delta(1,j')\bar{G}_{n-1}. \quad (6.54)$$

We may also employ the relation

$$e^{-W}f\left(\frac{\delta}{\delta U}\right)e^{W}=f\left(\frac{\delta}{\delta U}+\frac{\delta W}{\delta U}\right) \quad (6.55)$$

to express (6.53) as

$$\bar{G}_{n}=\frac{\text{perm}_{(n)}}{\det_{(n)}}\bar{G}_{1}{}^{0}\left[i,\,j';\,U+iv\frac{\delta}{\delta U}+iv\frac{\delta W}{\delta U}\right]. \quad (6.56)$$

This form leads to the differential characterization

$$\prod_{j=1}^{n}\mathfrak{F}_{j}\bar{G}_{n}=1_{n}, \quad (6.57)$$

where

$$\mathfrak{F}=(\bar{G}_{1}{}^{0})^{-1}-U-iv\delta/\delta U-iv\delta W/\delta U,$$
$$1_{n}=\text{perm}_{(n)} \quad \text{or} \quad \det_{(n)}\delta(j,j'). \quad (6.58)$$

In particular, the one-particle function satisfies

$$\mathfrak{F}_{1}\bar{G}_{1}[1;1';U]=\delta(1,1'). \quad (6.59)$$

Approximate solutions to these equations may be determined by methods other than those discussed in Sec. V. For example, it is sometimes convenient to separate the effects of direct interaction from those due to exchange and correlation. The former appear in the equation for $\bar{G}_{1}$ through the average local potential associated with the density,

$$(iv\delta W/\delta U)(1)=\pm i\!\int\! v(1,2)\bar{G}_{1}(2;2^{+}); \quad$$

the latter, constituting the difference between the direct effect and $\int iv(1,2)\bar{G}_{2}(12;1'2^{+})$, arise from $(iv\delta\bar{G}_{1}/\delta U)(1)$. We may introduce an effective local potential, V',

$$V'(1)=U(1)+i\left(v\frac{\delta W}{\delta U}\right)(1), \quad (6.60)$$

and rewrite (6.59) as

$$\left[(\bar{G}_{1}{}^{0})^{-1}(1)-V'(1)-i\!\int\! v(1;2)\frac{\delta V'(3)}{\delta U(2)}\frac{\delta}{\delta V'(3)}\right]$$
$$\times\bar{G}[1;1';V']=\delta(1,1'), \quad (6.61)$$

where we regard $\bar{G}$ as a functional of V' rather than U. Together with the equation for

$$\mathcal{K}'(1,2)\equiv\delta V'(1)/\delta U(2),$$

$$\mathcal{K}'(1,2)=\delta(1,2)\pm i\!\int\! v(1,3)\mathcal{K}'(4,2)\frac{\delta}{\delta V'(4)}\bar{G}_{1}(3;3^{+}),$$
$$\quad (6.62)$$

{which is related to the density correlation function

since we also have

$$\mathcal{K}'(1,2)=\delta(1,2)+i\!\int\! v(1,3)[\bar{G}(23;2^{+}3^{+})$$
$$-\bar{G}(2;2^{+})\bar{G}(3;3^{+})],\} \quad (6.63)$$

Eq. (6.61) determines the single-particle Green's function and $\mathcal{K}'$. The latter represents the change in potential produced by an alteration of the external potential, together with the induced change in the direct interaction potential. It may therefore be loosely described as an inverse dielectric constant. In a first approximation, which sets

$$\delta\bar{G}_{1}{}^{-1}(1;1')/\delta V'(2)=-\delta(1,1')\delta(1,2),$$

$\mathcal{K}'$ satisfies

$$\mathcal{K}'(1,2)\cong\delta(1,2)\pm i\!\int\! v(1,3)\mathcal{K}'(4,2)$$
$$\times\bar{G}_{1}(3;4)\bar{G}_{1}(4;3), \quad (6.64)$$

and $\bar{G}$ correspondingly satisfies

$$[(\bar{G}_{1}{}^{0}(1))^{-1}-V'(1)]\bar{G}_{1}(1;1')-i\!\int\! v(1,2)\bar{G}_{1}(1;3)$$
$$\times\mathcal{K}'(2,3)\bar{G}_{1}(3;1')=\delta(1,1'). \quad (6.65)$$

This approximation is useful for treating the high-density electron gas.[9] In particular, a knowledge of $\mathcal{K}'-1$ at equal times, which we denote by $(\mathcal{K}'-1)(0)$, serves to characterize the dependence of W on the interaction strength and exhibits its linear dependence on the total volume, (V):

$$\int v(1,2)\left[\frac{\delta W}{\delta v(1,2)}-\frac{i}{2}\bar{G}(1;1^{+})\bar{G}(2;2^{+})\right]$$
$$=\tfrac{1}{2}\int v(1,2)[\bar{G}(12;1^{+}2^{+})-\bar{G}(1;1^{+})\bar{G}(2;2^{+})]$$
$$=\tau(V)(\mathcal{K}'-1)(0). \quad (6.66)$$

The modifications in the interactions characterized by $\mathcal{K}$ occur in the expressions for all the Green's functions. So do terms like those discussed below (6.14) which describe the effect of the medium on propagation over long periods of time. One method for exhibiting these features is to introduce an effective potential $V[U]$ of the type encountered in Sec. V:

$$[(\bar{G}_{1}{}^{0})^{-1}-V]\bar{G}_{1}=1. \quad (6.67)$$

While V' is inappropriate to the discussion of strong short-range interactions for which $(\delta W/\delta U)\bar{G}_{1}$ and $\delta\bar{G}_{1}/\delta U$ are separately large and compensating, the effective potential V is generally applicable. Associated

[9] K. Sawada *et al.*, Phys. Rev. **108**, 507 (1957); **106**, 372 (1957). M. Gell-Mann and K. A. Brueckner, Phys. Rev. **106**, 364 (1957).

with V is a response function $\mathcal{K}$,

$$\mathcal{K}(1,1';2)=\frac{\delta V(1,1')}{\delta U(2)}=-\frac{\delta \bar{G}_1^{-1}(1;1')}{\delta U(2)}, \quad (6.68)$$

which also satisfies the equation

$$\delta \bar{G}_1(11';U)/\delta U(2)=[\bar{G}_1\mathcal{K}(\ ;2)\bar{G}_1](1;1')$$
$$=\pm[\bar{G}_2(12;1'2^+)-\bar{G}_1(1;1')\bar{G}_1(2;2^+)], \quad (6.69)$$

where the unspecified indices of $\mathcal{K}$ are understood to fit into the matrix multiplication. This response function is explicitly given by

$$\mathcal{K}(1,1';2)=\delta(1,2)\delta(1,1')\pm i\int v(1,3)\frac{\delta}{\delta U(2)}$$
$$\times[\bar{G}_2(13;1''3)\bar{G}_1^{-1}(1'';1')], \quad (6.70)$$

and may be determined by approximately solving the equation analogous to (6.62),

$$\mathcal{K}(1,1';2)-\delta(1,2)\delta(1,1')=\pm i\int v(1,3)\mathcal{K}(4,4';2)$$
$$\times\delta/\delta V(4,4')[\bar{G}_2(13;1''3)\bar{G}_1^{-1}(1'';1')]. \quad (6.71)$$

In the weak-correlation approximation we may make the replacement

$$\int d1''\bar{G}_2(13;1''3)\bar{G}_1^{-1}(1'';1')$$
$$\cong\delta(1,1')\bar{G}_1(3;3)\pm\bar{G}_1(1;3)\delta(3,1'), \quad (6.72)$$

and utilize the exact relation

$$\frac{\delta}{\delta V(4,4')}\bar{G}_1(1;2)=\bar{G}_1(1;4)\bar{G}_1(4';2), \quad (6.73)$$

thus obtaining

$$\mathcal{K}(1,1';2)\cong\delta(1,2)\delta(1,1')\pm i\int v(1,3)\bar{G}_1(3;4)$$
$$\times\bar{G}_1(4';3)\mathcal{K}(4,4';2)\delta(1,1')$$
$$+i\int v(1,1')\bar{G}_1(1;4)\bar{G}_1(4';1')$$
$$\times\mathcal{K}(4,4';2). \quad (6.74)$$

The equations for the Green's functions may be written in terms of $\mathcal{K}$ by regarding them as functionals of V rather than U. We first introduce the identity

$$\mathfrak{F}X=\mathfrak{F}\bar{G}_1\bar{G}_1^{-1}X=\left[1-i\int v(\ ,a)\bar{G}_1\frac{\delta}{\delta U(a)}\right]\bar{G}_1^{-1}X \quad (6.75)$$

for arbitrary X. If we set $X=1$, we obtain

$$V=U+iv\frac{\delta W}{\delta U}+i\int v(\ ,a)\bar{G}_1\mathcal{K}(\ ;a), \quad (6.76)$$

or more explicitly

$$V(1,1')=\left[U(1)+i\int v(1,2)\frac{\delta W}{\delta U(2)}\right]\delta(1,1')$$
$$+i\int v(1,2)\bar{G}_1(1;1'')\mathcal{K}(1'',1';2). \quad (6.77)$$

By applying (6.75) to (6.57) we obtain

$$\prod_{i=1}^{n}\left\{\left[1-i\int v(\ ,a)\bar{G}_1\frac{\delta}{\delta U(a)}\right]\bar{G}_1^{-1}\right\}_j \bar{G}_n[U]=1_n. \quad (6.78)$$

We shall henceforth adopt a summation convention for repeated literal indices and omit the integration sign in expressions like (6.78).

The equation for $\bar{G}_2$ is then given by

$$\{[1-iv(\ ,a)\bar{G}_1\delta/\delta U(a)]\bar{G}_1^{-1}\}_1$$
$$\times\{[1-iv(\ ,b)\bar{G}_1\delta/\delta U(b)]\bar{G}_1^{-1}\}_2\bar{G}_2[U]=1_{12}, \quad (6.79)$$

and is reduced to

$$(\bar{G}_1^{-1})_1\{[1-iv(\ ,b)\bar{G}_1\delta/\delta U(b)]\bar{G}_1^{-1}\}_2\bar{G}_2[U]=1_{12} \quad (6.80)$$

on multiplication by the inverse of the functional derivative expression for particle 1. When the corresponding derivative expression for the second particle is moved to the left, (6.80) becomes

$$\{[1-iv(\ ,b)\bar{G}_1\delta/\delta U(b)]_2(\bar{G}_1^{-1})_2(\bar{G}_1^{-1})_1$$
$$+iv(2,b)[\delta(\bar{G}_1^{-1})_1/\delta U(b)]\}\bar{G}_2[U]=1_{12}, \quad (6.81)$$

which, by the argument applied to (6.79), is equal to

$$\{(\bar{G}_1^{-1})_1(\bar{G}_1^{-1})_2-[1-iv(\ ,b)\bar{G}_1\delta/\delta U(b)]_2^{-1}$$
$$\times iv(2,a)[\mathcal{K}(11';a)]_1\}\bar{G}_2[U]=1_{12}. \quad (6.82)$$

To approximate the exact (6.82) it is convenient to prove an identity which we shall apply to the interaction term. We note that (6.75) implies

$$\mathfrak{F}^{-1}X=\bar{G}_1[1-iv(\ ,b)\bar{G}_1\delta/\delta U(b)]^{-1}X. \quad (6.83)$$

If we set $X=V'$ we obtain an equivalent expression for the combination

$$\mathfrak{F}^{-1}V'(a)\equiv[(\bar{G}_1^0)^{-1}-V'-iv\delta/\delta U]^{-1}V'(a) \quad (6.84)$$
$$=[(\bar{G}_1^0)^{-1}-V'-iv\delta/\delta U]^{-1}$$
$$\times[V'(a)+iv(a,b)\delta/\delta U(b)], \quad (6.85)$$

which may be written as

$$\mathfrak{F}^{-1}V'(a)=[V'(a)+iv(a,b)\delta/\delta U(b)]\bar{G}_1, \quad (6.86)$$

in the virtue of the commutativity of the two operators in (6.85).

The identity obtained from (6.83) and (6.86),

$$[V'(a)+iv(a,b)\delta/\delta U(b)]\bar{G}_1$$
$$=\bar{G}_1[1-iv(\ ,b)\bar{G}_1\delta/\delta U(b)]^{-1}V'(a) \quad (6.87)$$

may be reduced to

$$iv(a,b)\delta\bar{G}_1/\delta U(b)=iv(a,b)\bar{G}_1\mathcal{K}(\ ;b)\bar{G}_1$$
$$=\bar{G}_1[1-iv(\ ,c)\bar{G}_1\delta/\delta U(c)]^{-1}$$
$$\times iv(\ ,b)\bar{G}_1\delta V'(a)/\delta U(b) \quad (6.88)$$

by subtracting $V'(a)\bar{G}_1$, so that

$$[1-iv(\ ,c)\bar{G}_1\delta/\delta U(c)]^{-1}iv(\ ,b)\bar{G}_1\mathcal{K}(a,b)$$
$$=iv(a,b)\mathcal{K}(\ ;b)\bar{G}_1. \quad (6.89)$$

An equation for $\bar{G}_2$ is obtained from (6.82) and (6.89) by two approximations. The first involves neglecting the dependence on U of the function $\mathcal{K}'$, in (6.89), and leads to

$$[1-iv(\ ,c)\bar{G}_1\delta/\delta U(c)]^{-1}iv(\ ,a)\bar{G}_1$$
$$\cong i\mathcal{K}'^{-1}(a,c)v(c,b)\mathcal{K}(\ ;b)\bar{G}_1. \quad (6.90)$$

We describe by v the symmetric combination

$$v(a,b)=\mathcal{K}'^{-1}(a,c)v(c,b) \quad (6.91)$$

occurring in (6.90), and prove its asserted symmetry by observing that

$$\int v(2,4)\mathcal{K}'(1,4)=v(1,2)+i\int v(1,3)v(2,4)\frac{\delta^2 W}{\delta U(3)\delta U(4)}$$
$$=\int v(1,4)\mathcal{K}(2,4). \quad (6.92)$$

The matrix equivalent of this identity, $\mathcal{K}'v=v\mathcal{K}'^T$, implies this symmetry since

$$v=\mathcal{K}'^{-1}v=v(\mathcal{K}'^T)^{-1}=v(\mathcal{K}'^{-1})^T=(\mathcal{K}'^{-1}v)^T=v^T. \quad (6.93)$$

The second approximation involves replacing the interaction term in (6.82),

$$I\bar{G}_2\equiv[1-iv(\ ,b)\bar{G}_1\delta/\delta U(b)]_2^{-1}iv(\ ,a)_2$$
$$\times\mathcal{K}(\ ;a)_1\bar{G}_2[U], \quad (6.94)$$

by

$$I\bar{G}_2\cong\{[1-iv(\ ,b)\bar{G}_1\delta/\delta U(b)]^{-1}iv(\ ,a)\bar{G}_1\}_2$$
$$\times\mathcal{K}(\ ;a)_1(\bar{G}_1^{-1})_2\bar{G}_2, \quad (6.95)$$

in which the differential operator acts only on the bracketed expression, thus ignoring the U dependence associated with the other particle.

From (6.90) and (6.95) we obtain an approximate symmetrical interaction kernel,

$$I\cong i\mathcal{K}(\ ;a)_1v(a,b)\mathcal{K}(\ ;b)_2, \quad (6.96)$$

and an equation for $\bar{G}_2$,

$$\bar{G}_2\cong(\bar{G}_1)_1(\bar{G}_1)_2 1_{12}+i(\bar{G}_1)_1(\bar{G}_1)_2 I\bar{G}_2, \quad (6.97)$$

which indicates more explicitly than (5.56) modifying effects of the medium on the interaction characteristics of particles.